Part II Course: Astrophysics
Theory of Relativity
Michaelmas Term 2000: 24 Lectures, G. Efstathiou

Lecture One

1.1 Introduction

Our aim in developing relativity is to develop a mathematical way of expression physical laws. We need, therefore, to find physical principles to help us formulate physical laws mathematically, and we also need the mathematical machinery itself. Our requirements in developing relativity are:

• State physical laws in a way which is invariant under **certain** coordinate transformations.

• Develop a coordinate free language to express physical laws.

The above two points are central to the development of Special and General Relativity:

Special Relativity:

• Deals with coordinate transformations between inertial frames.

• Coordinate transformations are kinematical.

General Relativity

• Deals with coordinate transformations between non-inertial frames.

• Coordinate transformations dynamical.

• Need equations to describe the dynamics, so need a dynamical theory.

Evidently General Relativity ('GR'), as its name implies, is more general than Special Relativity ('SR'). The mathematics required for GR are quite complicated and the steps towards the full dynamical theory are non-trivial. In these lectures, I will assume that you have met special relativity before. I will therefore go through special relativity quite quickly, but emphasising those aspects of the theory that are most helpful for the subsequent development of GR. Although you will have met SR before, in my experience, students often have more difficulties with SR problems sets than with the superficially more sophisticated (but often rather mechanical) GR problem sets. Kinematic calculations in SR seem to be a particular difficulty. Do not underestimate the conceptual difficulties associated with SR. If you do not understand SR, then you do not really understand GR, even if you can solve the GR problems.

Most books on SR begin with some sort of description of the Michelson-Morley experiment and then introduce the Lorentz transformation. In fact, Einstein claimed that he wasn't influenced by this experiment. This is disputed by various historians of science and biographers of Einstein. One might think that these scholars are on strong ground, especially given that the experiment is referred to (albeit obliquely) in Einstein's papers. I think that it worth taking Einstein's claim at face value. Remember that Einstein was a *theorist* – one of the greatest theorists that has ever lived – and he had a *theorists* way of looking at physics. A good theorist develops an intuition about how Nature works that helps them formulate physical laws, for example, what symmetries might apply, which quantities are conserved, and so on. We can get a strong clue about Einstein's thinking from the *title* of his famous 1905 paper on Special Relativity. The first paragraph is reproduced here:

> ON THE ELECTRODYNAMICS OF MOVING BODIES
> By A. EINSTEIN
>
> It is known that Maxwell's electrodynamics – as usually understood at the present time – when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise – assuming equality of relative motion in the two cases discussed – to electric currents of the same path and intensity as those produced by the electric forces in the former case.

You see that Einstein's paper is not called 'Transformations between Inertial Frames', or 'A Theory in which the Speed of Light is Assumed to be a Universal Constant'. Electrodynamics is at the heart of Einstein's thinking – Einstein realized that Maxwell's equations of electromagnetism *required* Special Relativity.

Here are Maxwell's equations:

$$\left.\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{D} &= \rho \\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\ \boldsymbol{\nabla}\wedge\mathbf{E} &= -\tfrac{\partial\mathbf{B}}{\partial t} \\ \boldsymbol{\nabla}\wedge\mathbf{H} &= \mathbf{J}+\tfrac{\partial\mathbf{D}}{\partial t} \end{aligned}\right\} \begin{aligned} \mathbf{D} &= \epsilon\epsilon_0\mathbf{E} \\ \mathbf{B} &= \mu_0(\mathbf{H}+\mathbf{M}) \end{aligned} \quad ,$$

In free space we can set $\mathbf{J}=\rho=0$ and we then get the more obviously symmetrical looking equations,

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{D} &= 0\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\
\boldsymbol{\nabla}\wedge\mathbf{E} &= -\tfrac{\partial\mathbf{B}}{\partial t}\\
\boldsymbol{\nabla}\wedge\mathbf{H} &= \tfrac{\partial\mathbf{D}}{\partial t} \quad .
\end{aligned}
$$

Taking the curl of, say, the third equation and applying the relation

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \boldsymbol{\nabla}^2\mathbf{E} \quad ,$$

we derive the equations for electromagnetic waves

$$\nabla^2\mathbf{E} = \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \quad ,$$

and similarly

$$\nabla^2\mathbf{B} = \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}.$$

These have the form of a wave equation with a propagation speed c

$$
\begin{aligned}
\nabla^2\mathbf{E} &= \tfrac{1}{c^2}\ \tfrac{\partial^2\mathbf{E}}{\partial t^2}\\
c &= \tfrac{1}{\sqrt{\mu_0\epsilon_0}} \quad .
\end{aligned}
$$

Now the constants μ_0 and ϵ_0 are properties of the 'vacuum',

$\mu_0 =$ permeability of a vacuum $4\pi\times 10^{-7}$ Hm^{-1},

$\epsilon_0 =$ permittivity of a vacuum 8.85×10^{-12} Fm^{-1}.

This relation between the constants ϵ_0 and μ_0 and the speed of light was one of the most startling consequences of Maxwell's theory. But what do we mean by a 'vacuum'? Does it define an absolute frame of rest? If we deny any the existence of an absolute frame of rest, then how do we formulate a theory of electromagnetism? How do Maxwell's equations appear in frames moving with respect to each other? Do we change the value c – in which case what happens to the values of ϵ_0 and μ_0?

Einstein solves all of these problems at a stroke by saying that Maxwell's equations take the same mathematical form in all inertial frames.[1] The speed of light c is thus the same in all inertial frames. The theory of Special Relativity (including amazing conclusions such as $E = mc^2$) follows from a generalization of this simple and theoretically compelling assumption.

Maxwell's equations therefore require special relativity

You see that for a master theorist like Einstein, the Michelson-Morley experiment might well have been a side issue. Einstein could 'see' Special Relativity lurking in Maxwell's equation.

[1] I will discuss Maxwell's equations in detail in a later lecture

1.2 The Principles of Special Relativity

• **Definition of an Inertial Frame**

Is a frame in which spatial relations, as determined by rigid rods at rest in the frame are Euclidean, and in which there exists a universal time in terms of which free particles remain at rest or continue to move with constant speed along straight lines.

c.f. Newton's first law - Euclidean geometry is assumed and free particles are postulated to move in straight lines.

• **Axioms of Special Relativity** (Einstein)

1. The laws of physics are identical in all inertial frames.
2. There exists an inertial frame in which light signals in a vacuum always travel rectilinearly at constant speed c in all directions, independent of the motion of source.

Each of these axioms seems reasonable enough, but combine the two and one gets a surprising and apparently absurd conclusion:

Light signals in vacuum are propagated rectilinearly, with the same speed c at all times, in all directions, in all inertial frames.

Rindler calls this '*Einstein's law of light propagation*'.

1.3 The Lorentz Transformation

In an inertial frame, a free particle moves in a straight line.

Imagine a free falling clock (i.e., a physical system with repetitive events separated by equal increments of time τ).

Denote the coordinates of the clock by x^μ ($x^\mu \equiv t,\ x,\ y,\ z$). If the clock is falling freely, then we must have

$$\left.\begin{aligned} \tfrac{dx^\mu}{d\tau} &= \text{constant} \\ \tfrac{d^2x^\mu}{d\tau^2} &= 0 \end{aligned}\right\} \quad \text{clock follows linear trajectory.}$$

Denote the coordinates of the clock in some other inertial frame by x'^μ. The clock must, **by definition**, follow straight lines in any other **inertial** frame and so applying the chain rule of partial differentiation

$$\begin{aligned} \frac{dx'^\mu}{d\tau} &= \sum \tfrac{\partial x'^\mu}{\partial x^\nu}\tfrac{dx^\nu}{d\tau} = \text{constant}, \\ \frac{d^2x'^\mu}{d\tau^2} &= \sum \tfrac{\partial x'^\mu}{\partial x^\mu}\tfrac{d^2x^\nu}{d\tau^2} + \sum \tfrac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\sigma}\tfrac{dx^\nu}{d\tau}\tfrac{dx^\sigma}{d\tau} = 0. \end{aligned}$$

Hence, from the second of these equations we conclude that

$$\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\sigma} = 0 \quad ,$$

i.e. the transformation between the two inertial coordinate systems **must be linear**:

$$x'^\mu = \sum \Lambda^\mu_\nu x^\mu + B^\mu,$$

where the B^μ are (unimportant) constants and the Λ^μ_ν are a transformation matrix whose form we must uncover and which contains the physics of the theory.

Special Relativity is the study of those linear coordinate transformations between inertial frames that are consistent with Einstein's axioms.

1.3.1 The Galilean Transformation

Let us first look at the familiar **Galilean** (classical) transformation between two inertial frames S and S', with S' moving at speed v relative to S in the x-direction as shown in the figure.

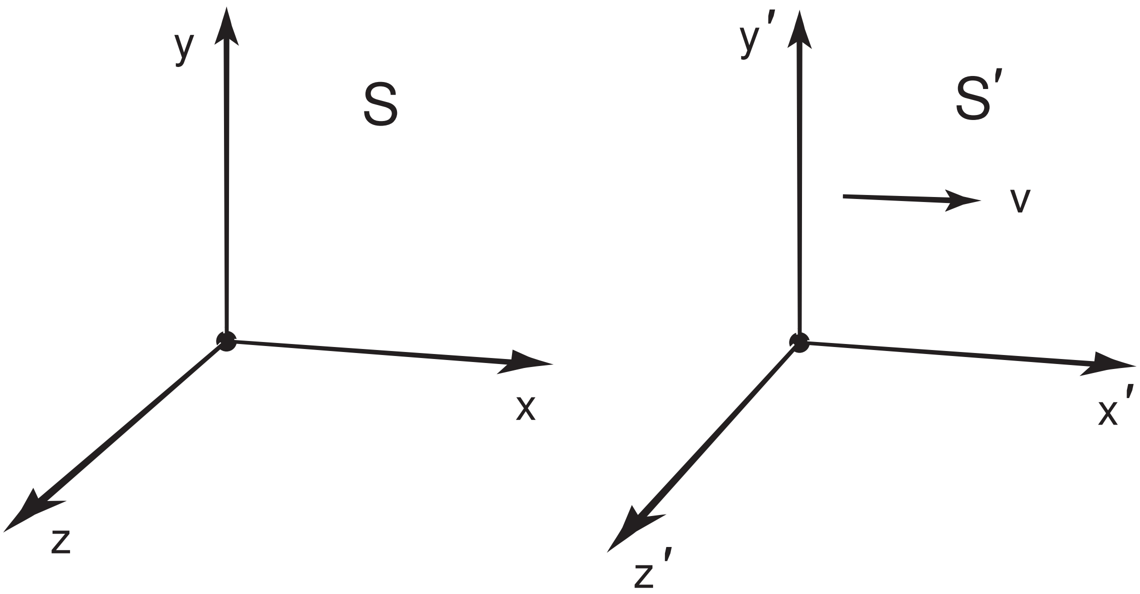

Figure 1:

The classical transformation for an event with coordinates (t, x, y, z) in S and (t', x', y', z') in S' is

$$\begin{aligned} t' &= t \\ x' &= x \quad -vt \\ y' &= y \\ z' &= z \end{aligned}$$

The first equation follows from the concept of a universal time, which is simply taken for granted. The last two equations are 'trivial' transformation equations, since the relative motion is in the x and x' direction. The third equation follows from the 'common sense' law of velocity addition, since differentiating with respect to $t' = t$ gives

$$u' = u - v.$$

Differentiating again shows that the accelerations are invariant,

$$f' = f.$$

1.3.2 Derivation of the Lorentz Transformation

We have shown that according to the principles of SR the coordinates of an event (t', x', y', z') in the inertial frame S' must be related to the coordinates (t, x, y, z) in S by a **linear transformation**. Now as with the Galilean transformation, we have two trivial transformations

$$y' = y,$$

$$z' = z.$$

(The motion is in the x-direction and we choose the zero points of the coordinate systems in the y and z directions to coincide). Now, define the x-coordinates so that $x' = 0$ corresponds to $x = vt$, then the general **linear relation** between the coordinates

$$x' = a_1 x + a_2 y + a_3 z + a_4 t + a_5,$$

must read

$$x' = \gamma(x - vt),$$

and, reversing v

$$x = \gamma(x' + vt').$$

If we now assume a 'universal time'

$$t' = t$$

then we must have

$$\begin{aligned} \gamma &= 1, \\ x' &= x - vt, \end{aligned}$$

and we recover the **Galilean** transformation.

But if we adopt Einstein's anzatz for light propagation (imagine a pulse of light emitted at $t = 0$ propogating from the origin $x = 0$)

$$\left.\begin{aligned} x &= ct \\ x' &= ct' \end{aligned}\right\} \qquad c = \text{constant},$$

then,

$$\left.\begin{aligned} ct' &= \gamma t(c-v), \\ ct &= \gamma t'(c+v), \end{aligned}\right\}$$

and so

$$\begin{aligned} c^2 &= \gamma^2(c^2-v^2), \\ \gamma &= \tfrac{1}{(1-v^2/c^2)^{1/2}} \quad . \end{aligned}$$

Hence, Einstein's law of light propagation leads to the Lorentz transformation,

$$\left.\begin{aligned} t' &= \gamma(t-vx/c^2), \\ x' &= \gamma(x-vt) \\ y' &= y \\ z' &= z \ . \end{aligned}\right\}$$

Are the two frames Euclidean?

In frame S, the propagation of light obeys

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0.$$

From the Lorentz transformation,

$$\begin{aligned} dt' &= \gamma dt - \tfrac{v\gamma}{c^2}dx, \\ dx' &= \gamma dx - v\gamma dt, \end{aligned}$$

and so,

$$\begin{aligned} c^2dt'^2 - dx'^2 &= c^2\gamma^2dt^2 + \tfrac{v^2\gamma^2}{c^2}dx^2 - 2v\gamma^2dxdt \\ &\quad -\gamma^2dx^2 - v^2\gamma^2dt^2 + 2v\gamma^2dxdt \\ &= c^2(1-v^2/c^2)\gamma^2dt^2 + (v^2/c^2-1)\gamma^2dx^2 \\ &= c^2dt^2 - dx^2 \quad . \end{aligned}$$

Hence

$$c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

therefore both frames are Euclidean.

1.4 A more mathematical description of Lorentz transformations

A Lorentz transformation is a transformation from one system of space-time coordinates x^α to another system x'^α, such that

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha \quad ,$$

where

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta},$$

$$\eta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta = 1, 2, 3, \\ -1 & \alpha = \beta = 0, \\ 0 & \alpha \neq \beta. \end{cases}$$

This is the unique, non-singular coordinate transformation that leaves the proper-time interval

$$d\tau^2 = dt^2 - d\mathbf{x}^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta$$

invariant.

Let us now prove this:

The transformations are linear, so

$$dx'^\alpha = \Lambda^\alpha_\gamma dx^\gamma$$

so,

$$\begin{aligned} \eta_{\alpha\beta} dx'^\alpha dx'^\beta &= \eta_{\alpha\beta} \Lambda^\alpha_\gamma dx^\gamma \Lambda^\beta_\delta dx^\delta \\ &= \eta_{\alpha\beta} \Lambda^\alpha_\gamma \Lambda^\beta_\delta dx^\gamma dx^\delta \\ &= \eta_{\gamma\delta} dx^\gamma dx^\delta \\ &= -d\tau^2 \quad , \end{aligned}$$

hence

$$d\tau'^2 = d\tau^2,$$

The set of all Lorentz transformations is called the *inhomogenous Lorentz group* or *Poincaré group*. Those with $a^\alpha = 0$ are called the the *homogenous Lorentz group*.

Those with $|\Lambda| = 1$, $\Lambda^0_0 \geq 1$, which can be converted to the identity δ^α_β by a continuous variation of parameters are called the *proper homogenous Lorentz group*.