Lecture Sixteen: Theory of Relativity

16.1 The Schwarzchild Singularity and Black Holes

Let's look at the Schwarzchild metric again

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \underbrace{\left[1 - \frac{2GM}{r}\right]^{-1}} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi, \tag{1}$$

What happens when $\frac{2GM}{r} = 1$?

Apparently, the metric blows up. The metric seems to diverge at the radius

$$\boxed{r_s = \frac{2GM}{c^2}.}$$

This characteristic radius is called the *Schwarzchild radius.* Now remember that we derived the Schwarzchild solution by solving the *vacuum* field equations

$$R_{\mu\nu} = 0.$$

The Schwarzchild radius for the Sun, is

$$r_s = \frac{2GM_\odot}{c^2} = 2.95 \text{ km},$$

and this is *much* smaller than the radius of the Sun ($R_\odot = 7 \times 10^5$ km). The Schwarzchild radius for a proton is

$$r_s = \frac{2GM_p}{c^2} = 10^{-50} \text{ cm},$$

again much smaller than the characteristic radius of a proton ($R_p = 10^{-13}$ cm). In fact for *most* real objects the Schwarzchild radius lies deep within the object where the vacuum field equations don't apply. You could simply argue that we should forget about the singularity in the Schwarzchild metric (1), since it occurs at a radius where the solution doesn't apply.

But *what if* there exist objects which are so compact that their radii lie well within the Schwarzchild radius? For such an object, the Schwarzchild solution looks very odd. Ignore the singularity in the metric (1) for the moment, and consider the situation shown in Figure 1. The region outside the Schwarzchild radius is denoted 'region I', and the region within the Schwarzchild radius is denoted 'region II'. Now look at the signs of the metric coefficients. In region I, the metric coefficients g_{00} and g_{rr} are both positive. We can therefore say that:

$$\text{In region I} \quad \begin{cases} \text{t} & \text{is timelike} \\ \\ \text{r} & \text{is spacelike} \end{cases} \quad \text{just as in Minkowski coordinates.}$$

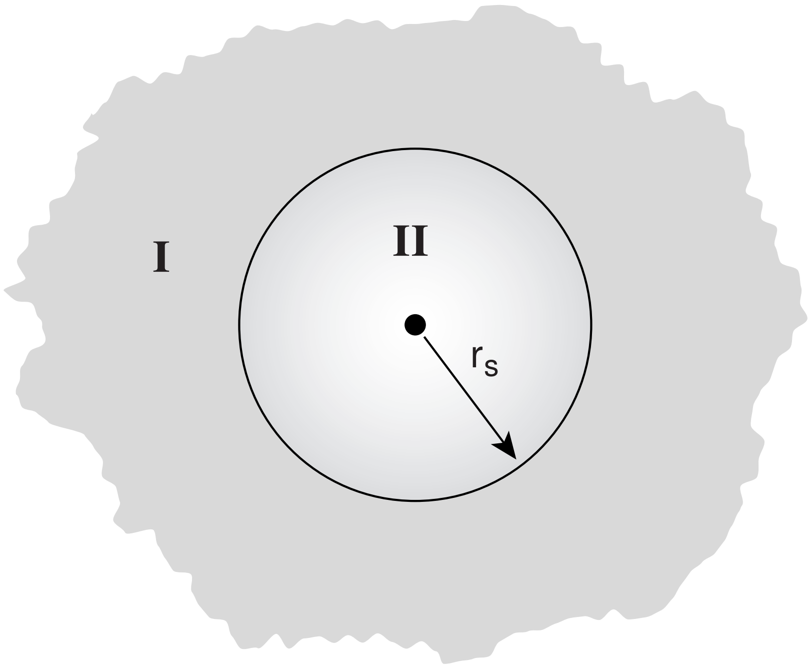

Figure 1: The regions interior and exterior to the Schwarzchild radius around a compact object.

But in region II, the metric coefficients g_{00} and g_{rr} change sign and so:

$$\text{In region II} \quad \begin{cases} t & \text{is spacelike} \\ & \\ r & \text{is timelike} \end{cases} \qquad r \text{ and } t \textit{ reverse character.}$$

This shows us that something significant does happen as we cross the boundary defined by the Scharzchild radius. Space and time seem to swap character. What does this mean? Is it physically significant?

Let's first deal with the singularity in the metric at $r = r_s$. In fact, there is nothing singular at $r = 2GM/c^2$. If you calculate the curvature tensor, you will find that it is regular everywhere except at $r = 0$ (where the curvature tensor does indeed diverge – this is the true singularity of the Schwarzchild solution). The singularity at $r = r_s$ in the Schwarzchild metric is just a *coordinate singularity.* To explain what I mean by a coordinate singularity, consider the metric of two-dimensional Euclidean space

$$ds^2 = dx^2 + dy^2.$$

Introduce a new coordinate

$$\xi = \frac{1}{3}x^3, \qquad d\xi = (3\xi)^{2/3}\, dx.$$

The metric of two-dimensional Euclidean space becomes

$$ds^2 = \frac{1}{(3\xi)^{4/3}} d\xi^2 + dy^2,$$

and so we seem to have a singularity at $\xi = 0$, *but this is simply a result of our choice of coordinates.* There is, of course, no 'real' singularity in the metric – the space is flat and the curvature tensor is zero everywhere.

In an analogous way, the coordinate singularity of the Schwarzchild metric is simply a result of the coordinate system that we have chosen to use. We can remove the singularity by making a clever transformation of coordinates. For example, in so called Kruskal coordinates

$$(t', r', \theta, \phi),$$

the coordinates t' and r' are related to our variables r and t by the relations

$$r'^2 - t'^2 = T^2 \left(\frac{r}{2\mu} - 1 \right) \exp \left(\frac{r}{2\mu} \right),$$

$$\frac{2r't'}{r'^2 + t'^2} = \tanh \left(\frac{t}{2\mu} \right),$$

where T is a constant. With these transformations, the coordinate singularity in the Schwarzchild metric disappears,

$$ds^2 = -\left(\frac{32\mu^3}{rT^2} \right) \exp \left(\frac{-r}{2\mu} \right) \left(dt'^2 - dr'^2 \right) - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,$$

and the metric is perfectly regular as long as r^2 is well defined and positive definite, *i.e.*,

$$r'^2 > t'^2 - T^2.$$

I will discuss Kruskal coordinates and the topology of this metric later. This coordinate transformation might seem quite contrived, but I will show you the motivation behind this particular transformation. My main point here is that it is possible to remove the coordinate singularity in the Schwarzchild metric by an appropriate change of coordinates.

16.2 Spacetime diagrams and the Schwarzchild solution

In discussing special relativity, we introduced space-time diagrams, like the one shown below.

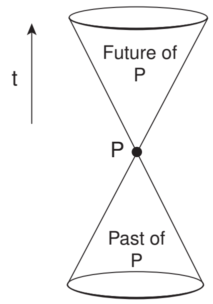

Figure 2: Space-time diagram in Special Relativity showing the light cone around event P.

In this Section, I want to investigate space-time diagrams of the Schwarzchild solution. I will write the metric as

$$ds^2 = (1 - 2\mu/r)\, dt^2 - \frac{dr^2}{(1 - 2\mu/r)} - r^2 d\Omega^2, \tag{2}$$

where $d\Omega$ is an element of solid angle. I have written the metric (2) in this form because I will usually ignore the angular coordinates in drawing space-time diagrams, *i.e.* the diagrams will show projections in the $r - t$ plane. The first thing to do is to analyse the light cone structure, *i.e.* to find the paths of incoming and outgoing radial light rays. From the metric (2) an ingoing radial light ray follows the path

$$\frac{dt}{dr} = -\frac{r}{(r - 2\mu)}. \tag{3a}$$

I have chosen the negative solution in (3a) because we must have $dr/dt < 0$ for an *ingoing* signal. Integrating (3a) we have

$$t = -r - 2\mu \ln|r - 2\mu| + \text{constant}, \quad \text{(ingoing light ray)}. \tag{3b}$$

For an outgoing signal select the positive solution

$$\frac{dt}{dr} = \frac{r}{(r - 2\mu)}, \tag{3c}$$

since we must have $dr/dt > 0$ for an *outgoing* signal. Integrating (3c)

$$t = r + 2\mu \ln|r - 2\mu| + \text{constant}, \quad \text{(outgoing light ray)}. \tag{3d}$$

The light cone structure is shown below:

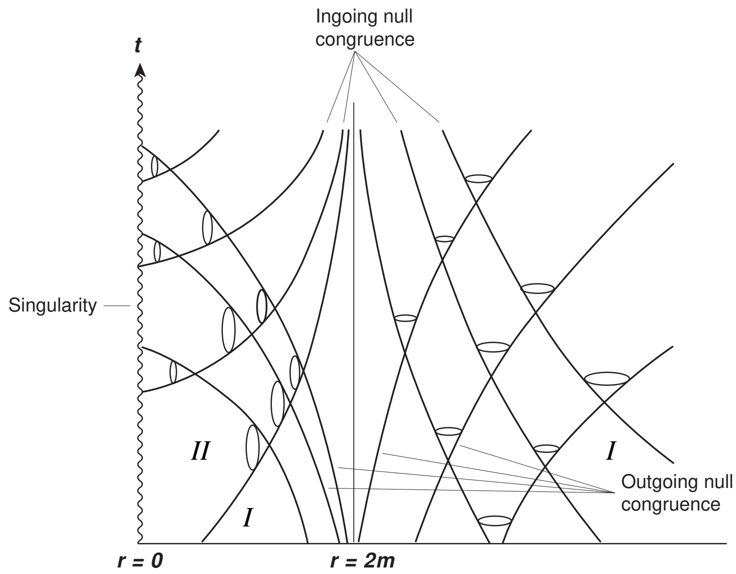

Figure 3: Light cone structure of the Schwarzchild solution.

Figure 3 looks pretty strange and so requires some words of explanation. The regions labelled I and II are as in Figure 1. Region I lies outside the Schwarzchild radius and region II lies inside the Schwarzchild radius. At large radii in region I, the gravitational field becomes weak and the metric tends to the Minkowski metric of Special Relativity. As expected, the light cone structure becomes the same as that of Special Relativity: incoming and outgoing light rays define lines of slope ± 1 in the diagram. As we approach the Schwarzchild radius, the ingoing light rays tend to $t \to +\infty$ and outgoing light rays tend to $t \to -\infty$. You might think that this means that it takes an infinite amount of time for an incoming signal to cross the Schwarzchild radius, but this would be wrong as I will show in the next Section. In region II the light cones flip orientation by 90°. This is a consequence of the reversal of time and space within this region that I pointed out in Section 16.1. The light cone structure is important because it tells us about the *causal structure* of the space-time. All incoming light rays *must* end at $r = 0$. But at $r = 0$ there is a *real* singularity – the curvature of the Schwarzchild solution diverges at $r = 0$. Furthermore, you can see that the future of any particle trajectory must end at the singularity. The conclusion is that *once you have crossed the Schwarzchild radius you must necessarily end up at a spacelike singularity at* $r = 0$. If you cross the Schwarzchild radius, you are therefore doomed, you will be sucked in to the singularity and you can never escape. To escape would require a violation of causality.

The causal structure in Figure 3 is clear enough, but it is not obvious from this diagram whether a particle does indeed cross the Schwarzchild radius. I will analyse this problem next.

16.3 Can a particle cross the boundary $r = 2\mu$?

The trajectories of freely falling particles in a gravitational field are described by the geodesic equations of motion. So, to answer the question of whether a particle can cross the Schwarzchild radius, we can solve the geodesic equations of motion for a radially infalling particle. This is quite an interesting problem because it will help you to gain an understanding of the physical meaning of coordinates in GR.

From the mathematical point of there is nothing new in this problem – we have already derived the geodesic equations of motion for the Schwarzchild metric in Section 14. There, we showed that

$$\frac{dt}{d\tau} = \frac{A}{1 - 2\mu/r}$$

where A is a constant (equation 5a of Section 14). As $r \longrightarrow \infty$, the Schwarzchild metric is of Minkowski form and so we can fix the constant A in terms of the radial speed u_0 of the particle at some large distance r.

$$d\tau^2 = dt^2 - dr^2 = dt^2(1 - u_0^2) = dt^2/\gamma_0^2,$$

hence we have determined the constant A,

$$\frac{dt}{d\tau} = \frac{\gamma_0}{(1 - 2\mu/r)}. \tag{4}$$

From the Schwarzchild metric (remember we are considering only radial motion)

$$ds^2 = d\tau^2 = (1 - 2\mu/r)\, dt^2 - (1 - 2\mu/r)^{-1}\, dr^2, \tag{5}$$

and so

$$(1-2\mu/r)\left(\frac{dt}{d\tau}\right)^2-(1-2\mu/r)^{-1}\left(\frac{dr}{d\tau}\right)^2=1. \tag{6}$$

Substituting for $dt/d\tau$ from (4) we find

$$\left(\frac{dr}{d\tau}\right)^2=\gamma_0^2-(1-2\mu/r)\,,$$

and so the equation of motion for the radial coordinate is

$$\left(\frac{dr}{dt}\right)^2=(1-2\mu/r)^2\,\frac{1}{\gamma_0^2}\left[\gamma_0^2-(1-2\mu/r)\right]=(1-2\mu/r)^2\left[1-\frac{1}{\gamma_0^2}\,(1-2\mu/r)\right]. \tag{7}$$

Now refering back to the metric (5), a *stationary* observer at the radial distance r measures time intervals,

$$dt'=(1-2\mu/r)^{1/2}\,dt,$$

and radial distances

$$dr'=(1-2\mu/r)^{-1/2}\,dr,$$

and so the stationary observer would assign the infalling particle a velocity

$$\frac{dr'}{dt'}=(1-2\mu/r)^{-1}\,\frac{dr}{dt}=\left[1-\frac{1}{\gamma_0^2}\,(1-2\mu/r)\right]^{1/2}. \tag{8}$$

This equation contains a surprising result. In the limit $r\to 2\mu$, $dr'/dt'\to 1$, *i.e.* as the particle approaches the Schwarzchild radius the stationary observer sees its velocity tending to the speed of light!

Let's now solve for the orbit of a radially infalling partice, assuming that it starts at rest at a radial distance $r=r_0$. From equation (6), its motion is described by

$$\frac{dr}{d\tau}=-\sqrt{\frac{2\mu}{r}},$$

where I have selected the negative solution to describe an infalling particle. Integrating this equation,

$$\begin{aligned}\tau &= \mathrm{K}-\frac{2}{3}\left(\frac{r^3}{2\mu}\right)^{1/3}, \qquad \text{where }\ \mathrm{K}\ =\ \text{constant},\\ &= \frac{2}{3}\left(\frac{r_0^3}{2\mu}\right)^{1/2}-\frac{2}{3}\left(\frac{r^3}{2\mu}\right)^{1/2}.\end{aligned}$$

Alternatively, we can describe the orbit by integrating equation (7),

$$\frac{dr}{dt}=-\,(1-2\mu/r)\sqrt{\frac{2\mu}{r}},$$

giving

$$\begin{aligned}
\int dt &= -\int \frac{dr}{(1-2\mu/r)\sqrt{2\mu/r}}, \\
t &= \frac{2}{3}\left[\left(\frac{r_0^3}{2\mu}\right)^{1/2} - \left(\frac{r^3}{2\mu}\right)^{1/2}\right] + 4\mu\left[\left(\frac{r_\circ}{2\mu}\right)^{1/2} - \left(\frac{r}{2\mu}\right)^{1/2}\right] \\
&+ \; 2\mu \ln\left\{\left[\frac{(r/2\mu)^{1/2}+1}{(r/2\mu)^{1/2}-1}\right]\left[\frac{(r_0/2\mu)^{1/2}-1}{(r_0/2\mu)^{1/2}+1}\right]\right\}.
\end{aligned}$$

From these solutions you can see that

$$\begin{aligned}
\tau &\longrightarrow \frac{2}{3}\left(\frac{r_\circ^3}{2\mu}\right)^{1/2}, \qquad \text{as} \quad r \to 0, \\
t &\longrightarrow \infty, \qquad\qquad\qquad \text{as} \quad r \to 2\mu.
\end{aligned}$$

How do we interpret these results. The two solutions are sketched below:

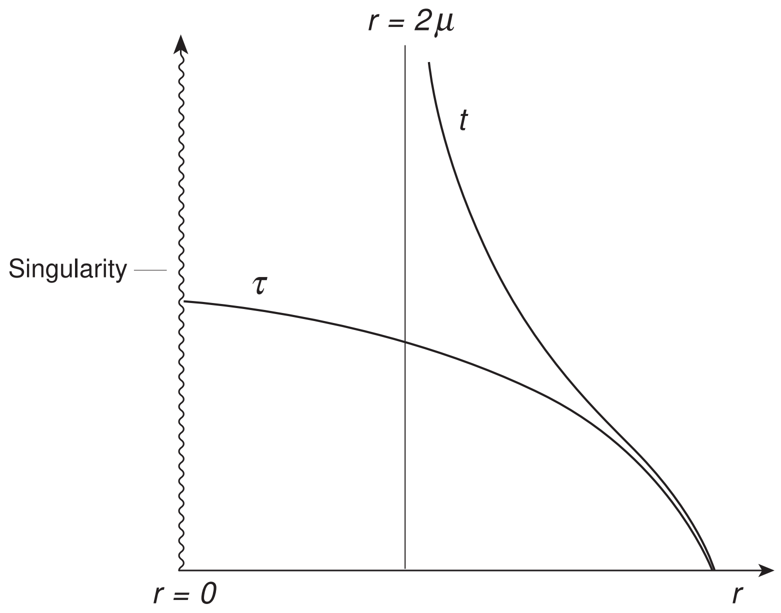

Figure 4: Orbit of a radially infalling particle

• t is the time measured by an observer at *infinity*. From such an observers point of view it takes an infinite amount of time for the particle to reach $r = 2\mu$.

• τ is the time measured by a clock at rest with respect to the particle. Evidently, it takes a *finite* time for the particle to reach singularity at $r = 0$.

The answer to the question of this Section is obvious from Figure 4. The particle *does* cross the Schwarzchild radius. Nothing strange happens at the Schwarzchild radius, the particle follows a continuous trajectory that crosses the Schwarzchild radius and ends up at the singularity. From the point of view of a distant observer, however, the particle takes an infinite amount of time to reach the Schwarzchild radius.

16.4 Eddington-Finkelstein coordinates

The space time diagram of Figure 3 is a bit confusing because of the discontinuities in the ingoing and outgoing light rays at the Schwarzchild radius. If we are discussing infalling particles, it would be nice to use coordinates in which ingoing radial null geodesics are *continuous* across $r = 2\mu$. Recall that ingoing radial light rays obey

$$t = -r - 2\mu \ln|r - 2\mu|,$$

so, transform to a new time variable

$$t' = \begin{cases} t + 2\mu \ln(r - 2\mu), & r > 2\mu, \\ t + 2\mu \ln(2\mu - r), & r < 2\mu, \end{cases}$$

then

$$t' = -r,$$

and

$$dt' = dt + \frac{2\mu}{(r - 2\mu)} dr,$$

and so the Schwarzchild metric looks like

$$ds^2 = (1 - 2\mu/r) c^2 dt'^2 + \frac{4\mu c}{r} dt' dr - (1 + 2\mu/r) dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

These coordinates are called *Eddington-Finkelstein* coordinates. The space-time diagram of the Schwarzchild solution in these coordinates is shown below.

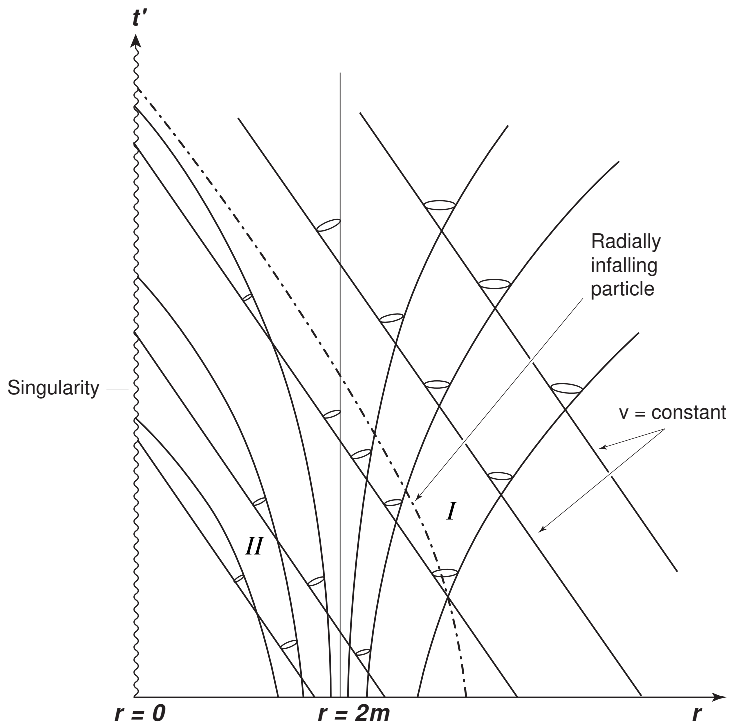

Figure 5: Light cone structure in Eddington-Finkelstein coordinates

By construction, the coordinate transformation has *straightened out* the ingoing radial null geodesics, so that they become lines of slope $-45°$ in the diagram. It is straightforward now to see that the radial trajectory of an infalling particle or photon is continuous at the Schwarzchild radius $r = 2\mu$. However, the light cone structure changes at the Schwarzchild radius. As you can see from the diagram, once you have crossed the boundary $r = 2\mu$, your future is directed towards the singularity. The Schwarzchild radius $r = 2GM/c^2$ defines an *event horizon* – a boundary of no return. Once a particle crosses the event horizon it must fall to the singularity at $r = 0$. A compact object that has an event horizon is called a *black hole.*

16.5 What happens if you fall towards a black hole?

Recall the equations of geodesic deviation that we derived in Section 10.1:

$$\frac{D^2\delta x^\mu}{D\tau^2} + R^\mu_{\lambda\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\rho = 0.$$

Now, if we calculate $R^\mu_{\lambda\sigma\rho}$ for the Schwarzchild solution, we find that

$$\frac{D^2\delta x^r}{D\tau^2} = \frac{2\mu c^2}{r^3}\delta x^r \qquad \longleftarrow \quad \text{positive, so stretching in the radial direction}$$

$$\left.\begin{aligned}\frac{D^2\delta x^\theta}{D\tau^2} &= -\frac{\mu c^2}{r^3}\delta x^\theta \\ \frac{D^2\delta x^\phi}{D\tau^2} &= -\frac{\mu c^2}{r^3}\delta x^\phi\end{aligned}\right\} \longleftarrow \quad \text{negative, so compression in the } \theta \text{ and } \phi \text{ directions}$$

So the equations of geodesic deviation tell show that you would be stretched out like spaghetti – like the poor astronaut shown below.

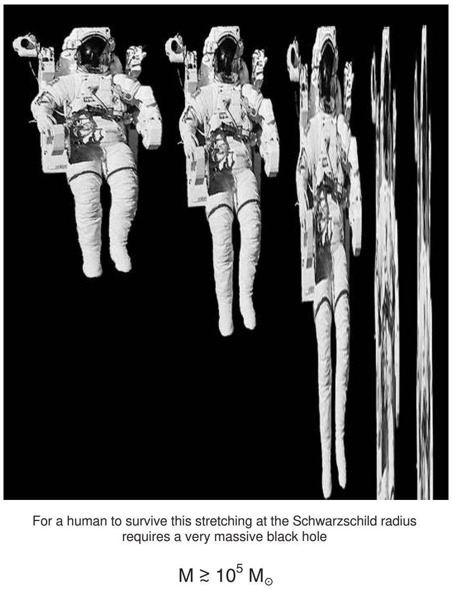

Figure 6: Astronaut stretched by the tidal field of a black hole.

For a human to survive this stretching at the Schwarzchild radius requires a very massive black hole with

$$M \gtrsim 10^5 M_\odot.$$

If you fell towards a supermassive black hole, with say $M \sim 10^9 M_\odot$ (as are believed to lie at the centres of some galaxies, see the next Section) you would cross the event horizon without feeling a thing. Everything would seem normal. But your fate will have been sealed – you will end up shredded by the tidal forces of the black hole as you approach the singularity from which there is no escape. If you fell towards a 'small' black hole of mass say $10 M_\odot$, you would be shredded apart by the tidal forces of the hole well before you reached the event horizon.