17.3 Stable and Unstable orbits around a Schwarzchild black hole

In Newtonian dynamics the equation of motion of a particle in a central potential is

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V(r) = E,$$

where $V(r)$ is an "effective potential". For an orbit around a point mass, the effective potential is

$$V(r) = \frac{h^2}{2r^2} - \frac{GM}{r},$$

where h is the specific angular momentum of the particle. The concept of an effective potential is, I am sure, familiar to you. If you sketch the effective potential, you can see easily that bound orbits have two turning points and that a circular orbit corresponds to the special case where the particle sits at the minimum of the effective potential.

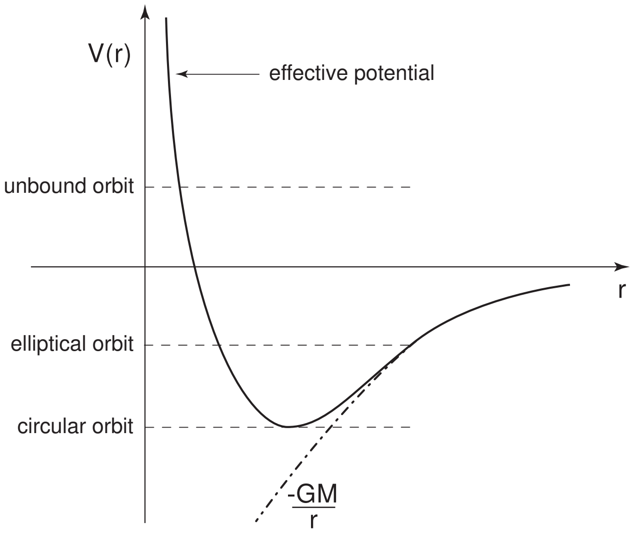

Figure 3: The Newtonian effective potential showing how an angular momentum barrier prevents particles reaching $r = 0$.

Furthermore, you can see that in Newtonian dynamics, a finite angular momentum provides an *angular momentum barrier* preventing a particle reaching $r = 0$. This is not true in General Relativity.

In GR, the equation of motion of a particle around a central mass point is

$$\beta(r)\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{\alpha(r)} = -E, \tag{1}$$

where

$$\alpha(r) = \frac{1}{\beta(r)} = \left(1 - \frac{2\mu}{r}\right).$$

Since $d\tau^2 = Edp^2$ we can rewrite equation (1) as

$$\left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{1}{E} \tag{2}$$

where $h = J/\sqrt{E}$ and $V(r)$ is an effective potential

$$V(r) = \left(1 - \frac{2\mu}{r}\right)\left(1 + \frac{h^2}{r^2}\right). \tag{3}$$

If we transform to the dimensionless variables

$$\begin{aligned} x &= \frac{2r}{r_s} = \frac{rc^2}{GM}, \\ h'^2 &= \frac{4h^2}{r_s^2}, \end{aligned}$$

we can write the effective potential as

$$V(x) = \left(1 - \frac{2}{x}\right)\left(1 + \frac{h'^2}{x^2}\right) = 1 - \frac{2}{x} - \frac{2h'^2}{x^3} + \frac{h'^2}{x^2}. \tag{4}$$

Differentiating this expression,

$$\frac{dV}{dx} = \frac{2}{x^2} + \frac{6h'^2}{x^4} - \frac{2h'^2}{x^3},$$

and so the extrema of the effective potential are located at the solutions of the quadratic equation

$$x^2 - h'^2 x + 3h'^2 = 0,$$

i.e. at

$$x = \frac{h'}{2}\left\{h' \pm \sqrt{h'^2 - 12}\right\}.$$

IF $h' = \sqrt{12} = 2\sqrt{3}$ *there is only one extremum*, and there are no turning points in the orbit for lower values of h'. Figure 4 shows the effective potential for several values of h'. The dots show the locations of stable circular orbits. The maxima in the potential are the locations of *unstable* circular orbits.

What is the physical significance of this result? The smallest stable circular orbit has

$$x_{\min} = 6, \quad i.e. \;\; r_{\min} = 6\frac{GM}{c^2}.$$

Gas in an accretion disc settles into circular orbits around the compact object as shown schematically in Figure 2. However, the gas slowly loses angular momentum because of turbulent viscosity (the turbulence is thought to be generated by magnetohydrodynamic instabilities). As the gas loses angular momentum it moves slowly towards the black hole, gaining gravitational potential energy and heating up. Eventually it loses enough angular momentum that it can no longer follow a stable circular orbit and so it falls into the black hole. We can, therefore, make a rough energy of the efficiency of energy radiation in an accretion disc. The maximum efficiency is of order the gravitational binding energy at the smallest stable circular orbit divided by the rest mass enery of the gas

$$\epsilon_{\rm acc} \approx \frac{1}{2}\frac{GMm}{r_{\min}}\frac{1}{mc^2} \simeq \frac{1}{12} \sim 8\%.$$

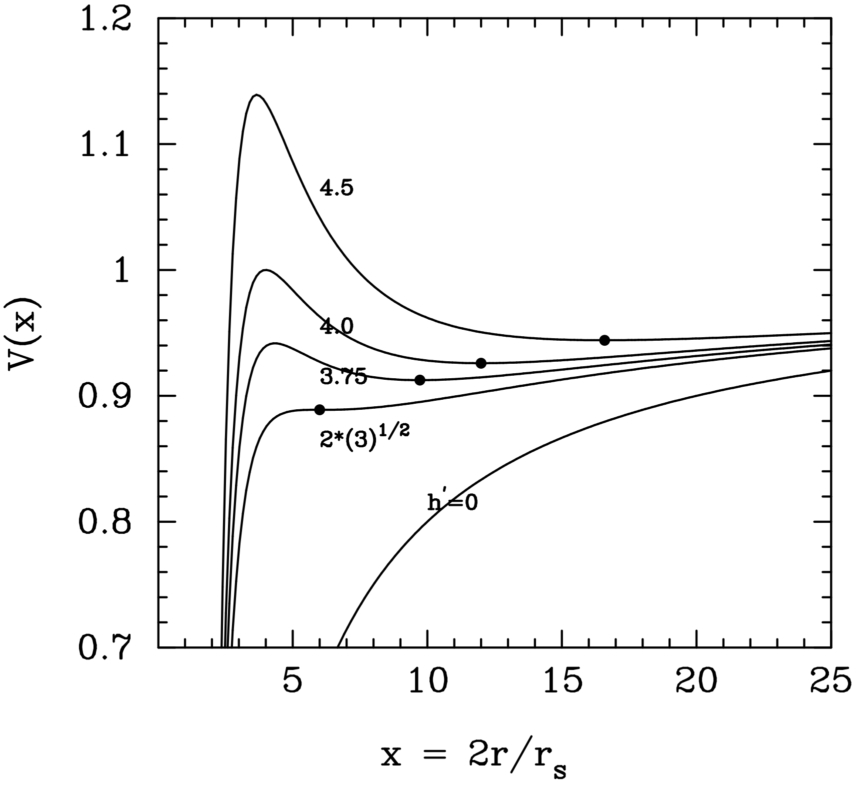

Figure 4: The effective potential (4) plotted for several values of the angular momentum parameter h'.

An accretion disc can convert perhaps a few percent of the rest mass energy of the gas into radiation. Compare this with the efficiency of nuclear burning of hydrogen to helium (26 MeV per He nucleus),

$$\epsilon_{\rm nuclear} \sim 0.7\%.$$

Accretion discs are capable of converting rest mass energy into radiation with an efficiency that is about 10 times greater than the efficiency of nuclear burning of hydrogen. The 'accretion power' of black holes cause some of the most energetic phenomena known in the Universe.

17.4 Supermassive black holes

The first quasar[1] (3C273) was discovered in 1963 by Maarten Schmidt. He measured a cosmological redshift of $z = 0.15$ for this object – unprecedently high at the time. (Quasars have since been discovered with redshifts as high as $z = 5.8$) Quasars are very luminous, typically 100– 1000 times brighter than a large galaxy. However, they are *compact*, so compact in fact that quasars look like stars in photographs. In fact, from variability and other studies one can infer that the size of the continuum emitting region of a quasar is of order a few parsecs or less. How can we explain such a phenomenon? Imagine an object

[1] *Quasi-stellar radio sources.* We now know that the majority of quasars are radio quiet, and so they are often called QSOs for *quasi-stellar object.*

radiating many times the luminosity of an entire galaxy from a region smaller than the Solar System. Donald Lynden-Bell was one of the first to suggest that the quasar phenomenon is caused by accretion of gas on to a *supermassive* black hole residing at the centre of a galaxy. The black hole masses required to explain the high luminosities of quasars are truly spectacular – we require black holes with a few million to a few billion times the mass of the Sun.

Do such supermassive black holes exist? The evidence in recent years has become extremely strong. Using the Hubble Space Telescope it is possible to probe the velocity dispersions of stars in the central regions of galaxies. According to Newtonian dynamics, we would expect the characteristic velocities to vary as

$$v^2 \sim \frac{GM}{r}. \tag{5}$$

If the central mass is dominated by a supermassive black hole, then we expect the typical velocities of stars to *increase* as we go closer to the centre. This is indeed what is found in a number of galaxies. From the rate of increase of the velocities with radius, we can estimate the mass of the central object which seems to be correlated with the mass of the bulge component of the galaxy:

$$M_{bh} \approx 0.006 M_{bulge}.$$

It seems as though at the time of galaxy formation, about half a percent of the mass of the bulge material collapses right to the very centre of a galaxy to form a supermassive black hole. During this phase the infalling gas radiates efficiently producing a quasar. When the gas supply is used up, the quasar quickly fades away leaving a dormant massive black hole that is starved of fuel. Nobody has yet developed a convincing theory of how this happens, or of what determined the masses of the central black holes.

Table 2: Supermassive Black Holes

		$M_{bh}/M_\odot$	Evidence
$***$	M87	2×10^9	stars & optical disc
$**$	NGC 3115	1×10^9	stars
$**$	NGC 4594 (Sombrero)	5×10^8	stars
$**$	NGC 3377	1×10^9	stars
$*****$	NGC 4258	4×10^7	masing H_20 disc
$**$	M31 (Andromeda)	3×10^7	stars
$**$	M32	3×10^6	stars
$****$	Galactic Centre	2.5×10^6	stars & 3D motions

A skeptic might argue that these observations merely prove that a dense compact object exists at the centres of galaxies, not necessarily a black hole. But there are two beautiful observational results that probe compact objects on parsec scales – making it almost certain that the central objects are black holes. In our own Milky Way Galaxy it is possible to measure the *proper motions* of stars in the Galactic centre (using infrared wavelengths to penetrate through the dense dust that obscures optical light). This has allowed astronomers to actually see the stars moving and so infer their three dimensional motions. These observations imply that there exists a black hole of mass $2.5 \times 10^6 M_\odot$ at the centre of our Galaxy.

In a remarkable set of observations, a disc of H_2O masars has been detected in the galaxy NGC 4258 using VLBI. The VLBI observations measure the velocities of the masing clouds on scales of $\sim 0.3 - 2$ parsec and are well fitted by a thin (actually slightly warped) disc in circular motion (See Figure 5). The mass of the central black hole is estimated to be $4 \times 10^7 M_\odot$.

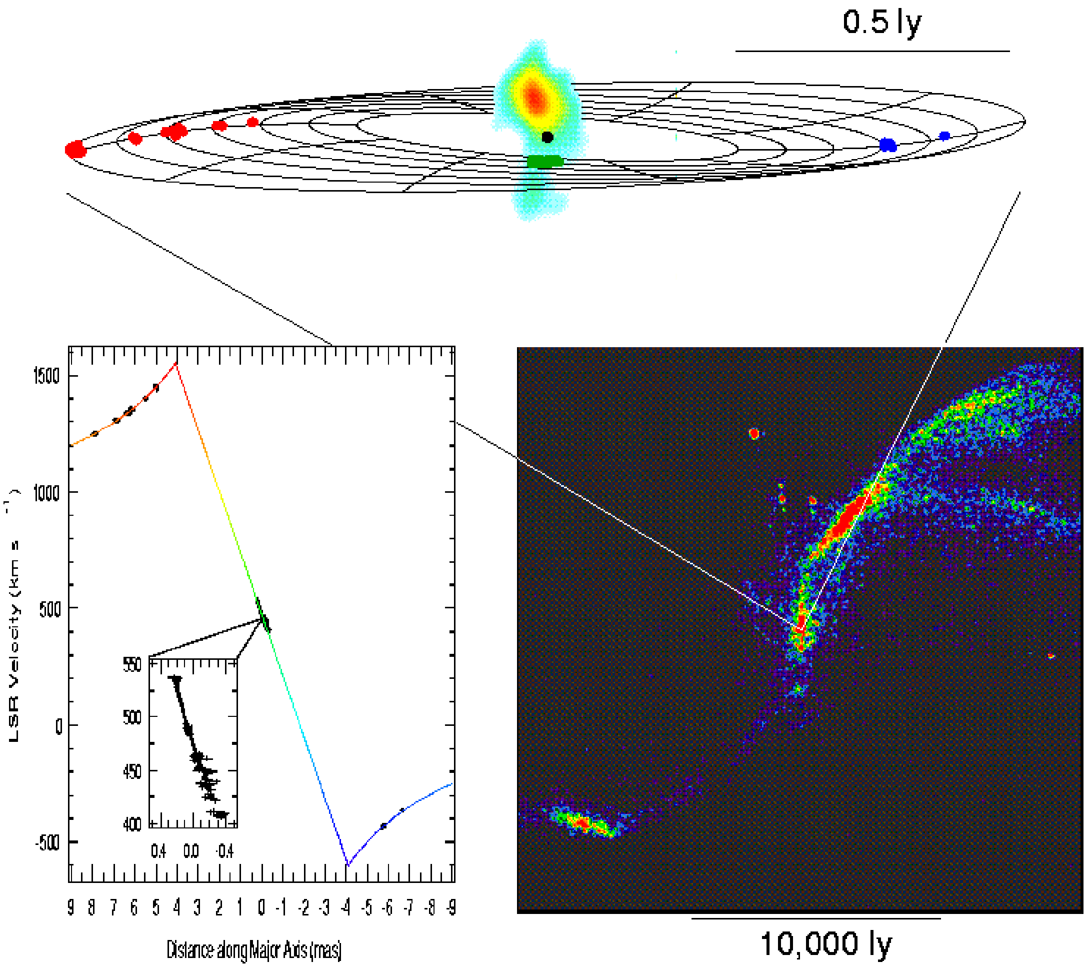

Figure 5: The masing H_20 disc in the centre of NGC4258.

Table 2 lists the masses of some supermassive black holes. I have given the observations a five star rating. The masing disc of NGC 4528 gets a full five stars – this is the strongest observational evidence for a supermassive black hole. The stellar motions in the Galactic Centre get four stars, though some astronomers might argue that this evidence is so strong that it should rate five stars. Most of the other observations are based on measurements of stellar velocity dispersions. This is fairly strong evidence, but not completely convincing[2]

[2]The interpretation of velocity dispersion measurements requires some assumptions about the degree of velocity anisotropy.

and so rate only two stars.

Finally, to end this section, I will show you some recent results from X-ray spectroscopy. The following picture shows spectral line of iron at X-ray wavelengths measured in the Seyfert galaxy MCG-6-30-15. The iron lines come from the inner parts of the accretion disc around the black hole and the line profile allows one to probe the strong gravitational regime. The lines are assymetric as a result of special relativistic beaming and general relativity. The detailed shape of the line profile depends on the metric and hence on whether the black hole is rotating (see the notes on the Kerr metric). The hope is that the line profiles can be measured and modelled in detail and so be used to infer the *angular momenta* of black holes as well as their masses.

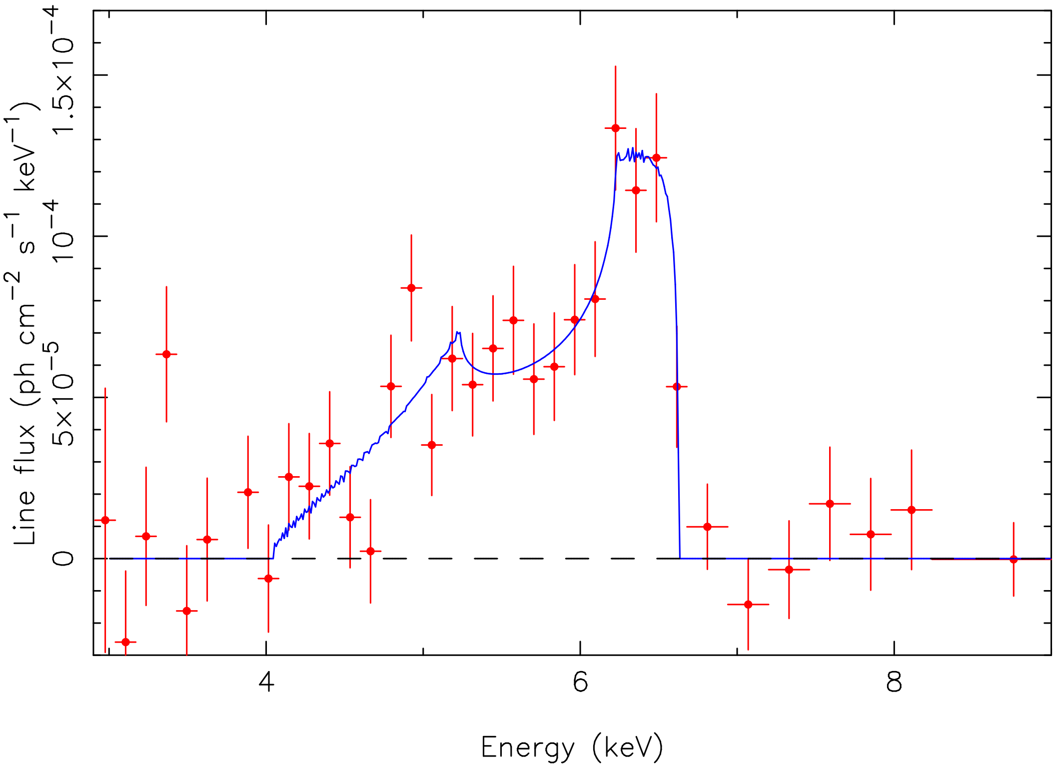

Figure 6: The line profile of iron K-alpha from MCG-6-30-15 observed by the ASCA satellite (Tanaka et al, 1995, Nature, 375, 659). The emission line is extremely broad, with a width indicating velocities of order one-third of the speed of light. There is a marked asymmetry towards energies lower than the rest-energy of the emission line (6.4 keV). This asymmetry is most likely caused by gravitational and relativistic-Doppler shifts near the black hole at the center of the galaxy. The solid line shows the model profile expected from a disk of matter orbiting the hole, extending between 3 and 10 Schwarzschild radii.