Part II Course: Astrophysics
Theory of Relativity
Michaelmas Term 2000: 16 Lectures, G. Efstathiou

Lecture Thirteen

Kerr Black Holes [1]

In 1963 Kerr discovered a remarkable solution of the vacuum field equations $R_{\mu\nu} = 0$.

In Schwarzchild–like coordinates (Boyer–Linquist coordinates) the metric looks like

$$ds^2 = \left.\begin{array}{l} \frac{\triangle}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 \\ -\frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\phi - adt\right]^2 \\ -\frac{\rho^2}{\triangle}dr^2 - \rho^2 d\theta^2 \end{array}\right\}$$

$$\left.\begin{array}{rcl} \triangle &=& r^2 - 2\mu + a^2 \\ \rho^2 &=& r^2 + a^2\cos^2\theta \end{array}\right\}$$

So, we have an additional parameter a.

In $\mathcal{L}ta \longrightarrow 0,\ \triangle \longrightarrow r^2(1-2\mu/r),\ \rho^2 \longrightarrow r^2$ and the Kerr metric tends to the standard Schwarzchild form

$$ds^2 \longrightarrow (1-2\mu/r)\,dt^2 - (1-2\mu/r)^{-1}\,dr^2 - r^2 d\Omega^2$$

The metric coefficients are independent of t and ϕ, so the solution is:-

- stationary
- axially symmetric

It is also invariant under the **joint** transformation

$$t \longrightarrow -t \ \ , \ \ \phi \longrightarrow -\phi$$

[1] This material is not examinable

suggesting that the metric has something to do with spin.

In fact, the light–cone structure (in particular, frame–dragging, see later) suggest that we make the following identifications

$$\begin{array}{rcl} \mu & \longrightarrow & \text{geometric mass} \\ \mu a & \longrightarrow & \text{angular momentum of the hole} \end{array}$$

Structure of a Kerr black hole

The Riemann curvature scalar $\mathcal{R}$ diverges when

$$\rho^2 = r^2 + a^2 \cos^2\theta = 0$$

$$\text{i.e., } r = 0, \quad \cos\theta = 0$$

The coordinates $r,\ \theta,\ \phi$ are not spherical polars, but are related to Cartesian–type coordinates by

$$\begin{aligned} x &= r\sin\theta\cos\phi + a\sin\theta\sin\phi \\ y &= r\sin\theta\sin\phi - a\sin\theta\cos\phi \\ z &= r\cos\theta \end{aligned}$$

So, the singularity is at

$$x^2 + y^2 = a^2, z = 0$$

i.e. it lies on a ring in the equatorial plane.

The g_{00} component of the metric is

$$g_{00} = \frac{\triangle}{\rho^2} - \frac{a^2\sin^2\theta}{\rho^2} = \frac{r^2 - 2\mu r + a^2\cos^2\theta}{\rho^2}$$

so surfaces of infinite redshift are given by

$$r_S \pm = \mu \pm \left(\mu^2 - a^2\cos^2\theta\right)^{1/2}$$

The surface S^- lies within S^+ and S^+ defines an ellipsoid with radii 2μ at the equator and $\mu + (\mu^2 - a^2)^{1/2}$ at the poles.

To find the event horizon s, locate the surface where $r =$ constant becomes null.

This is given by setting $g^{11} = 0$ [2]

$$g^{11} = -\frac{\triangle}{\rho^2} = -\frac{(r^2 - 2\mu r + a^2)}{(r^2 + a^2\cos^2\theta)} = 0$$

So there are two event horizons at

$$r = r\pm = \mu \pm \left(\mu^2 - a^2\right)^{1/2}$$

Can define 3 distinct regions

I $r+ < r < \infty$

II $r- < r < r_+$

III $0 < r < r_-$

insert figure 1

insert figure 2

Insert figure 3

insert figure 4

[2]We haven't proved this, but the proof is quite simple, see standard textbooks