Lecture Nineteen: Theory of Relativity[1]

19.1 Gravitational radiation

Imagine that we have a small perturbation to the metric of Special Relativity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \quad . \tag{1}$$

The Ricci tensor is, by definition,

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta}$$

, and so

$$R_{\mu\nu} \approx \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \mathcal{O}\left(h^2_{\mu\nu}\right). \tag{2}$$

The definition of the affine connection is

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\left\{\frac{\partial g_{\nu\rho}}{\partial x^{\mu}} + \frac{\partial g_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right\} \approx \frac{1}{2}\eta^{\rho\lambda}\left\{\frac{\partial h_{\rho\nu}}{\partial x^{\mu}} + \frac{\partial h_{\rho\mu}}{\partial x^{\nu}} - \frac{\partial h_{\mu\nu}}{\partial x^{\rho}}\right\} + \mathcal{O}\left(h^2_{\mu\nu}\right),$$

and to first order we can raise and lower the indices of $h_{\mu\nu}$ using $\eta^{\mu\nu}$, e.g.,

$$\eta^{\lambda\rho} h_{\rho\nu} = h^{\lambda}_{\nu}.$$

So,

$$\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}\eta^{\rho\lambda}\left[\frac{\partial h_{\rho\lambda}}{\partial x^{\mu}} + \underbrace{\frac{\partial h_{\rho\mu}}{\partial x^{\lambda}}}_{\searrow} - \underbrace{\frac{\partial h_{\mu\lambda}}{\partial x^{\rho}}}_{\swarrow}\right]$$

These terms cancel and so we have

$$\begin{aligned}\frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} &= \frac{1}{2} h^{\lambda}_{\lambda,\mu\nu},\\ \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} &= \frac{1}{2}\left[h^{\lambda}_{\nu,\mu\lambda} + h^{\lambda}_{\mu,\nu\lambda} - \eta^{\rho\lambda}\frac{\partial^2 h_{\mu\nu}}{\partial x^{\rho}\partial x^{\lambda}}\right].\end{aligned}$$

From equation (2) the Ricci tensor is therefore

$$R_{\mu\nu} = -\partial_{\lambda}\partial^{\lambda} h_{\mu\nu} - \frac{1}{2}h^{\lambda}_{\nu,\mu\lambda} - \frac{1}{2}h^{\lambda}_{\mu,\nu\lambda} + \frac{1}{2}h^{\lambda}_{\lambda,\mu\nu}.$$

Recall the definition of the d'Alembertian operator,

$$\eta^{\alpha\beta}\frac{\partial}{\partial x^{\beta}}\frac{\partial}{\partial x^{\alpha}} = \Box^2 \equiv \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2,$$

so, we can rewrite the expression for the Ricci tensor in terms of the d'Alembertian operator, and use the field equations in the form

$$R_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}\right) = -16\pi G S_{\mu\nu},$$

to derive the following wave equation,

$$\Box^2 h_{\mu\nu} + \frac{1}{2}h^{\lambda}_{\nu,\mu\lambda} + \frac{1}{2}h^{\lambda}_{\mu,\mu\nu} - \frac{1}{2}h^{\lambda}_{\lambda,\mu\nu} = 16\pi G S_{\mu\nu}. \tag{3}$$

[1]Most of the material in this section extends beyond the syllabus and is not examinable.

19.2 Gauge transformations

Remember the discussion of the electromagnetic field equations in Special Relativity. There, we derived a wave-like equation for a four-vector potential which we simplified by choosing a specific gauge (we adopted the Lorentz gauge). We can do a similar thing here. Suppose we perform an infinitesimal coordinate transformation,

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x). \tag{4}$$

The metric will change as

$$g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\rho}} g^{\lambda\rho}, \tag{5}$$

and so since to first order in h,

$$g^{\lambda\rho} \approx \eta^{\lambda\rho} - h^{\lambda\rho},$$

the metric (5) will change as

$$\begin{aligned} g'^{\mu\nu} &= \left(\delta^{\mu}_{\lambda} + \frac{\partial \epsilon^{\mu}}{\partial x^{\lambda}}\right)\left(\delta^{\nu}_{\rho} + \frac{\partial \epsilon^{\nu}}{\partial x^{\rho}}\right)\left(\eta^{\lambda\rho} - h^{\lambda\rho} + \mathcal{O}(h^2) + \dots\right. \\ &= \eta^{\mu\nu} + \delta^{\nu}_{\rho}\eta^{\lambda\rho}\frac{\partial \epsilon^{\mu}}{\partial x^{\lambda}} + \delta^{\mu}_{\lambda}\eta^{\lambda\rho}\frac{\partial \epsilon^{\nu}}{\partial x^{\rho}} - h^{\mu\nu} + \dots, \end{aligned}$$

i.e.,

$$h'^{\mu\nu} = h^{\mu\nu} - \eta^{\lambda\nu}\frac{\partial \epsilon^{\mu}}{\partial x^{\lambda}} - \eta^{\rho\mu}\frac{\partial \epsilon^{\nu}}{\partial x^{\rho}},$$

and lowering indices,

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} - \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}}. \tag{7}$$

It therefore follows that for any small coordinate change, ϵ_{μ} , the perturbation

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} - \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}}$$

will *also* be a solution of the gravitational field equations. (The physical content of the theory cannot be altered by simply relabelling the coordinates.) We can therefore choose ϵ_{μ} to simplify the wave equation for $h_{\mu\nu}$, without affecting any of the physics – just as we did when we chose the Lorentz gauge in electromagnetism. The freedom to add terms to $h_{\mu\nu}$ as in (7) is called a *gauge freedom*, and the choice of a particular form of ϵ_{μ} fixes a specific gauge.

An obvious choice is to choose ϵ_{μ} so that

$$\frac{\partial h^{\mu}_{\nu}}{\partial x^{\mu}} = \frac{1}{2}\frac{\partial h^{\mu}_{\mu}}{\partial x^{\nu}}.$$

This requires that

$$\Box^2 \epsilon_{\nu} = \frac{\partial h^{\mu}_{\nu}}{\partial x^{\mu}} - \frac{1}{2}\frac{\partial h^{\mu}_{\mu}}{\partial x^{\nu}},$$

and the wave equation for the $h_{\mu\nu}$ can be written as two equations:

$$\begin{aligned} \Box^2 h_{\mu\nu} &= 16\pi G S_{\mu\nu}, \\ \frac{\partial h^{\mu}_{\nu}}{\partial x^{\mu}} &= \frac{1}{2}\frac{\partial h^{\mu}_{\mu}}{\partial x^{\nu}}. \end{aligned}$$

This particular gauge choice is called the *harmonic gauge* and corresponds to $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$.

We can write the retarded solution to the wave equation straightfowardly,

$$h_{\mu\nu} = 4G \int d^3\mathbf{x}' \frac{S_{\mu\nu}\left(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|\right)}{|\mathbf{x} - \mathbf{x}'|}, \tag{8}$$

but it is exceedingly difficult to find exact solutions that satisfy these equations. Nevertheless, we can see qualitatively what is going on. If some irregular distribution of matter is wobbling as a function of time, it will induce small disturbances in the gravitational field, represented by the perturbation to the metric tensor $h_{\mu\nu}$. These disturbances in the gravitational field satisfy a wave equation and so propagate at the speed of light. Such disturbances are called *gravitational waves*. It is difficult to evaluate the integral in (8) for realistic sources, (for example, supernovae explosions, or binary black holes). The theory of gravitational radiation is quite a technical subject, often involving complex numerical computations. However, it is easy to describe the propagation of gravitational waves in a vacuum, as I will show in the next section.

19.3 Plane waves and polarization states

In vacuum, the equations describing gravitational waves in the harmonic gauge are

$$\Box^2 h_{\mu\nu} = 0, \tag{9a}$$

$$\frac{\partial h^{\mu}_{\nu}}{\partial x^{\mu}} = \frac{1}{2}\frac{\partial}{\partial x^{\nu}} h^{\mu}_{\mu}. \tag{9b}$$

These equations have plane wave solutions,

$$h_{\alpha\beta} = A_{\alpha\beta}\exp\left(ik_{\mu}x^{\mu}\right) + A^{*}_{\alpha\beta}\exp\left(-ik_{\mu}x^{\mu}\right), \tag{10a}$$

where the wavevector k_{μ} and the amplitudes $A_{\alpha\beta}$ must satisfy the conditions

$$k_{\mu}k^{\mu} = 0, \tag{10b}$$

$$k_{\mu}A^{\mu}_{\nu} = \frac{1}{2}k_{\nu}A^{\mu}_{\mu}. \tag{10c}$$

Since $h_{\mu\nu}$ is symmetric, the matrix $A_{\alpha\beta}$ must also be symmetric.

A 4×4 symmetric matrix has 10 independent components. Equation (10c) supplies 4 additional conditions reducing the number of independent components of $A_{\alpha\beta}$ to 6. However, only *two* of these components are physically significant. To see this, let's make an infinitesimal coordinate transformation,

$$\epsilon^{\mu}(x) = i\epsilon^{\mu}\left(ik_{\lambda}x^{\lambda}\right) - i\epsilon^{*\mu}\exp\left(-ik_{\lambda}\epsilon^{\lambda}\right).$$

We have already shown that

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial\epsilon_{\mu}}{\partial x^{\nu}} - \frac{\partial\epsilon_{\nu}}{\partial x^{\mu}},$$

is a solution of the field equations. So, forming the derivatives of ϵ_μ,

$$\frac{\partial \epsilon_\mu}{\partial x^\nu} = -k_\nu \epsilon_\mu \exp\left(ik_\lambda x^\lambda\right) - k_\nu \epsilon_\mu^* \exp\left(-ik_\lambda \epsilon^\lambda\right),$$

$$\frac{\partial \epsilon_\nu}{\partial x^\mu} = -k_\mu \epsilon_\nu \exp\left(ik_\lambda x^\lambda\right) - k_\mu \epsilon_\nu^* \exp\left(-i\kappa_\lambda \epsilon^\lambda\right),$$

we can see that

$$h'_{\mu\nu} = \left(A_{\mu\nu} + k_\nu \epsilon_\mu + \kappa_\mu \epsilon_\nu\right) \exp\left(ik_\lambda x^\lambda\right) + \left(A^*_{\mu\nu} + k_\nu \epsilon_\mu^* + k_\mu \epsilon_\nu^*\right) \exp\left(-ik_\lambda x^\lambda\right),$$

is a solution of the field equations. The coordinate transformation has therefore generated a new solution,

$$A'_{\mu\nu} = A_{\mu\nu} + k_\nu \epsilon_\mu + k_\mu \epsilon_\nu.$$

Now, suppose that we have a plane wave moving along the x-direction

$$\begin{aligned} k^2 &= k^3 = 0, \\ k_\mu k^\mu &= 0, \qquad \Rightarrow k^1 = k^0 = k, \end{aligned}$$

and applying the harmonic gauge condition

$$k_\mu A^\mu_\nu = \frac{1}{2} k_\nu A^\mu_\mu,$$

$$A^\mu_\nu = \eta^{\mu\rho} A_{\rho\nu}, \text{ so } \left\{ \begin{array}{lllllll} A^0_0 &=& \eta^{0\rho} A_{\rho 0} &=& \eta^{00} A_{00} &=& A_{00}, \\ A^0_i &=& \eta^{0\rho} A_{\rho i} &=& A_{0i}, && \\ A^j_0 &=& \eta^{j\rho} A_{\rho 0} &=& -A_{j0}, && \\ A^j_i &=& \eta^{j\rho} A_{\rho i} &=& -A_{ji}, && \end{array} \right.$$

hence

$$A^\mu_\mu = A^0_0 + A^i_i = A_{00} - A_{11} - A_{22} - A_{33}.$$

The harmonic gauge condition thus gives

$$\left\{ \begin{array}{l} A^0_0 + A^0_1 = A^1_0 + A^1_1 = \frac{1}{2}\left(A_{00} - A_{11} - A_{22} - A_{33}\right), \\ A^0_2 + A^1_2 = 0, \\ A^0_3 + A^1_3 = 0, \end{array} \right.$$

i.e.

$$\left\{ \begin{array}{ll} A_{00} + A_{01} = -A_{10} - A_{11} = \frac{1}{2}\left(A_{00} - A_{11} - A_{22} - A_{33}\right), & (11a) \\ A_{02} + A_{12} = 0, & (11b) \\ A_{03} + A_{31}. & (11c) \end{array} \right.$$

We can therefore express A_{i0} and A_{33} in terms of the other six components

$$\begin{array}{lll} A_{02} &=& -A_{12}, \qquad\qquad \text{from } (11b) \\ A_{03} &=& -A_{13}, \qquad\qquad \text{from } (11c) \\ A_{01} &=& -\frac{1}{2}(A_{00} + A_{11}), \qquad \text{from } (11a) \end{array}$$

and using (11a)

$$\begin{aligned} -\tfrac{1}{2} A_{11} &= \tfrac{1}{2}(A_{00} - A_{22} - A_{33}) + A_{10} \\ &= \tfrac{1}{2}(A_{00} - A_{22} - A_{33}) - \tfrac{1}{2}(A_{00} + A_{11}), \end{aligned}$$

hence,

$$A_{33} = -A_{22}.$$

Now perform an infinitesimal coordinate transformation

$$A'_{\mu\nu} = A_{\mu\nu} + k_\nu \epsilon_\mu + k_\mu \epsilon_\nu,$$

$$\begin{array}{rcl} A'_{00} & = & A_{00} + 2k\epsilon_0 \\ A'_{11} & = & A_{11} - 2k\epsilon_1 \\ A'_{12} & = & A_{12} - k\epsilon_2 \\ A'_{13} & = & A_{13} - k\epsilon_3 \\ A'_{33} & = & A_{33} \\ A'_{32} & = & A_{32} \end{array}$$

The first four terms *can be set to zero by a coordinate transformation*. Since physical effects are described by tensorial equations and must therefore be independent of the choice of coordinate system, the first four terms can produce *no physical effects*. We can therefore describe a plane gravitational wave using only two amplitudes,

$$A_{33} = -A_{22} \equiv -h_{22}, \qquad A_{32} \equiv h_{32},$$

or in matrix notation,

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{32} \\ 0 & 0 & h_{32} & -h_{22} \end{pmatrix}.$$

We can see what this means physically from the equations of geodesic deviation. In the weak field limit, these read

$$\begin{array}{rcl} \dfrac{D^2 y}{D\tau^2} & = & -\dfrac{1}{2}\left(h_{22,00}y + h_{23,00}z\right), \\ \dfrac{D^2 z}{D\tau^2} & = & -\dfrac{1}{2}\left(h_{23,00}y + h_{22,00}z\right). \end{array}$$

If we have only an h_{22} mode (which I will call + mode)

$$\frac{D^2 y}{D\tau^2} = -\frac{1}{2}h_{22,00}y, \qquad \frac{D^2 z}{D\tau^2} = \frac{1}{2}h_{22,00}z,$$

and the motion is as shown in Figure (1a). If we have only an h_{32} mode (which I will call the $\times$ mode),

$$\frac{D^2 y}{D\tau^2} = -\frac{1}{2}h_{32,00}z, \qquad \frac{D^2 z}{d\tau^2} = -\frac{1}{2}h_{32,00}y,$$

and the motion is as shown in Figure (1b).

These diagrams show the responses of test particles to the gravitational waves with two distinct polarization states (+ and $\times$ polarizations). Notice also that the combination

$$h_{22} \pm i h_{32},$$

transforms as $e^{\pm i2\theta}$ under spatial rotations. The polarization states of *light* transform as $e^{\pm i\theta}$. Photons therefore have a helicity of ± 1, whereas the mediating particles of gravity (gravitons) have helicity ± 2.

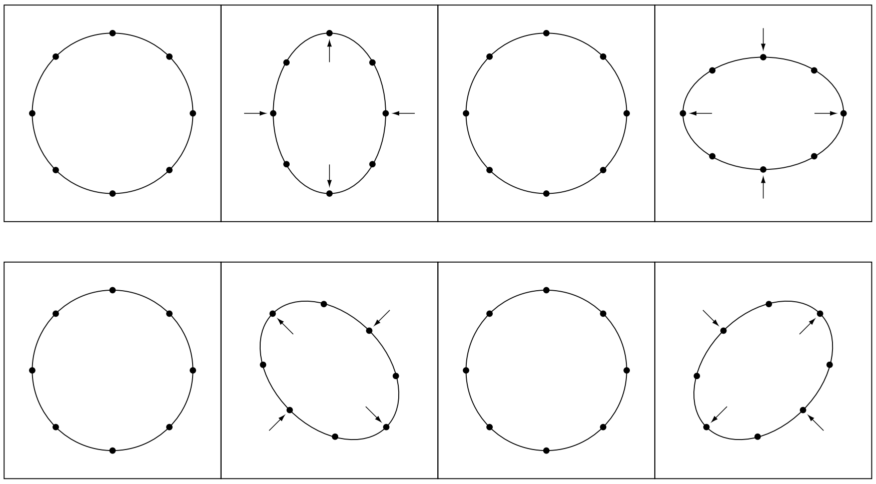

Figure 1: The upper panel (Figure 1a) shows the response of a ring of test particles to a gravitational wave with the + polarization mode. The lower panel (Figure 1b) shows the reponse to a wave with × polarization.

19.4 Detection of Gravitational Waves

What kind of sources might emit gravitational waves? Evidently some type of moving matter, but the matter distribution must be irregular. This follows from Birkhoff's theorem in General Relativity. Birkhoff's theorem is the relativistic equivalent of the Newtonian theorem that the external gravitational field of a spherical mass distribution is the same as that of a mass point at the origin. Birkhoff's theorem can be stated in a number of ways. One form is as follows:

> *"A spherically symmetric vacuum solution in the exterior region is necessarily static."*

This immediately tells us that a spherically symmetric pulsating star cannot generate gravitational waves. To generate gravitational waves, a mass distribution must have a time-varying quadrupole moment. For slowly moving near-Newtonian source, one can show that the rate of loss of energy from gravitational radiation is given by the *quadrupole formula*,

$$L_{\rm GW} = \frac{1}{5}\left\langle \dddot{I}^2 \right\rangle = \frac{1}{5}\left\langle \dddot{I}_{\alpha\beta}\dddot{I}^{\alpha\beta} \right\rangle,$$

where $I_{\alpha\beta}$ is the quadrupole moment

$$I_{\alpha\beta} = \int \rho \left(x_\alpha x_\beta - \frac{1}{3}\delta_{\alpha\beta} r^3 \right) d^3\mathbf{x}.$$

The following diagram shows the expected amplitudes for various sources of gravitational radiation. Detectors on Earth are sensitive only to high frequency gravitational wave (frequencies greater than a few Hz). Seismic noise on Earth prevents the detection of gravitational waves of lower frequency. Ground based detectors can be of two types:

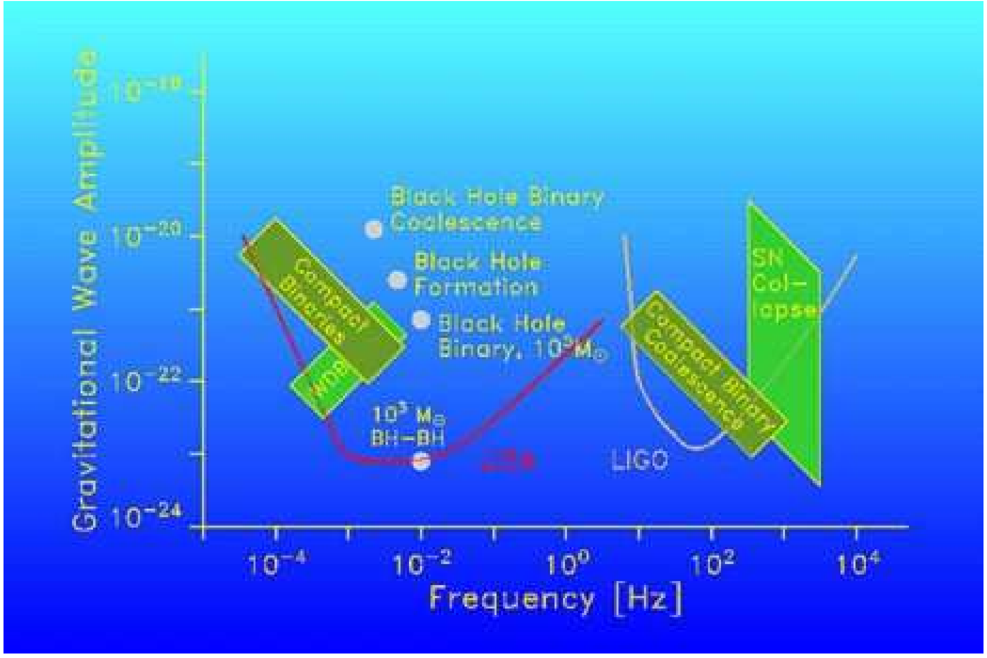

Figure 2: The expected amplitudes of various sources of gravitational radiation as a function of frequency.

- *bar detectors* – measuring the correlated distortion of a large gravitating mass.
- *laser interferometers* – measuring interferometrically the correlated distortion of multiple reflections of laser beams along vacuum paths of several kilometers or more. The most ambitious of these huge Michaelson laser interferometers is the US LIGO project (Laser Interferometer Gravitational Wave Observatory). This has two interferometers each with a 3km baseline. Two European detectors are also under construction.

Ground based detectors should eventually detect strong sources in our Galaxy, for example, the collapse of the stellar core following a supernova explosion. They should be capable of detecting the final stages of the merger of compact binary systems in external galaxies.

Laser interferometers in space should be able of detecting a wider range of sources. Figure 3 shows a schematic plot of the LISA mission (Laser Interferometer Space Antenna) which has recently been proposed as a joint ESA and NASA mission. This is a technically challenging mission and the launch date may well be later than 2015!

This exciting project is expected to detect gravitational waves at cosmological distances, for example, the mergers of massive binary black holes systems in distant galaxies.

19.5 The Binary pulsar

Gravitational radiation has already been discovered. In 1974, Joe Taylor and Russell Hulse discovered a remarkable binary pulsar system with the giant Aricebo radio telescope. (The

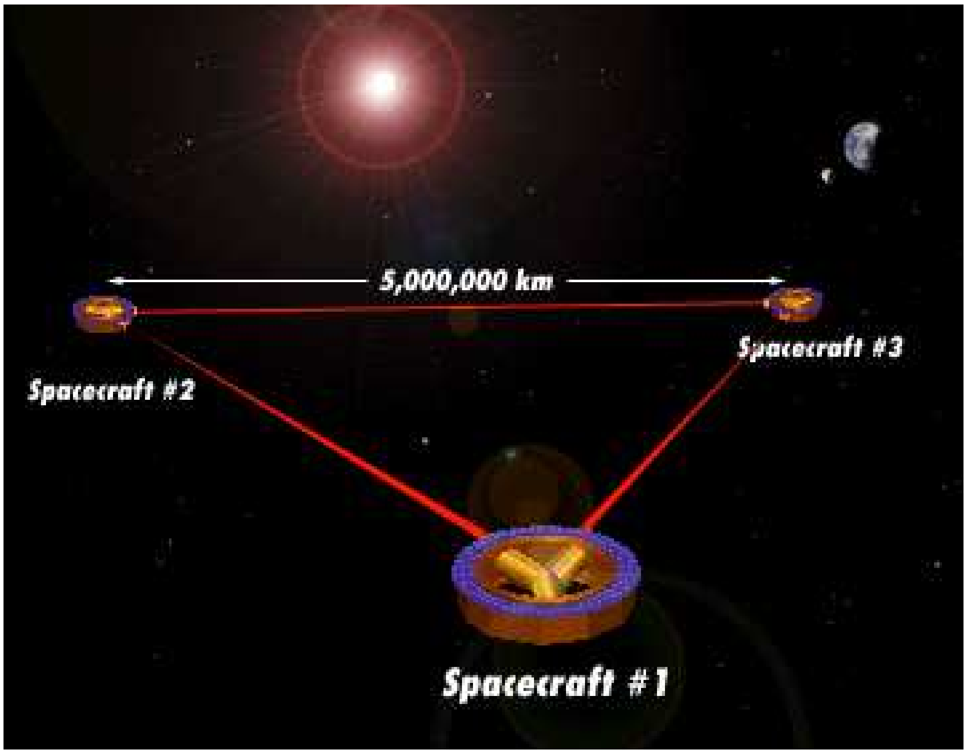

Figure 3: Schematic diagram of the planned ESA/NASA space mission LISA.

were awarded the Nobel prize in 1993 for this discovery.) This system (PSR 1913+16) has period of 59ms, but this period *changes* by 80μs with a period of 7.75 hours. Taylor and Hulse concluded that the pulsar is member of a compact binary system (see Figure 4). A long series of careful observations have established the parameters of this binary system with high precision. The masses of the two neutron stars and the inclination angle, eccentricity and period of the orbit are

$$\begin{aligned} M_1 &= 1.44117 M\odot, \\ M_2 &= 1.38747 M\odot, \\ \sin i &= 0.76, \\ e &= 0.617, \\ \tau_{\text{orbit}} &= 7.751939337\,\text{hr}. \end{aligned}$$

Most remarkably, Taylor and collaborators established that the orbital period is *gradually shrinking* at a rate of

$$\frac{1}{\tau_{\text{orbit}}} \frac{d\tau_{\text{orbit}}}{dt} = 2.4 \times 10^{-12} \text{ per year.}$$

This decline in the orbital period is exactly what is predicted from General Relativity. The orbit is shrinking because gravitational radiation is draining energy from the orbit. Figure 5 shows the accumulated shift in the time of periastron (*i.e.* the time of closest approach) over a period of nearly 20 years. The orbital decay rate is in very precise agreement with the predictions of GR. In fact, this observation not only proved the existence of gravitational radiation, but it also makes GR one of the most accuractly tested of all physical theories, including quantum electrodynamics!

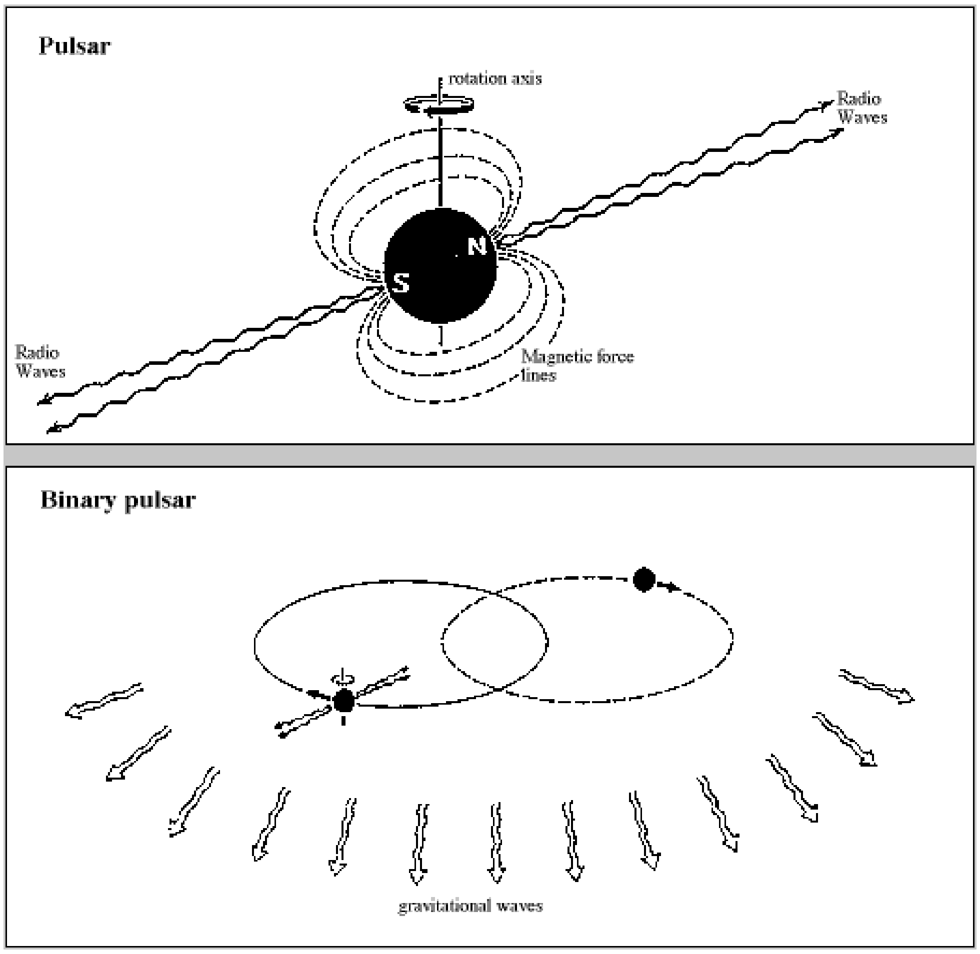

Figure 4: The upper picture shows a spinning pulsar emitting highly beamed radio waves along the magnetic poles. The lower panel shows a sketch of a binary pulsar system, slowly losing energy by emitting gravitational radiation.

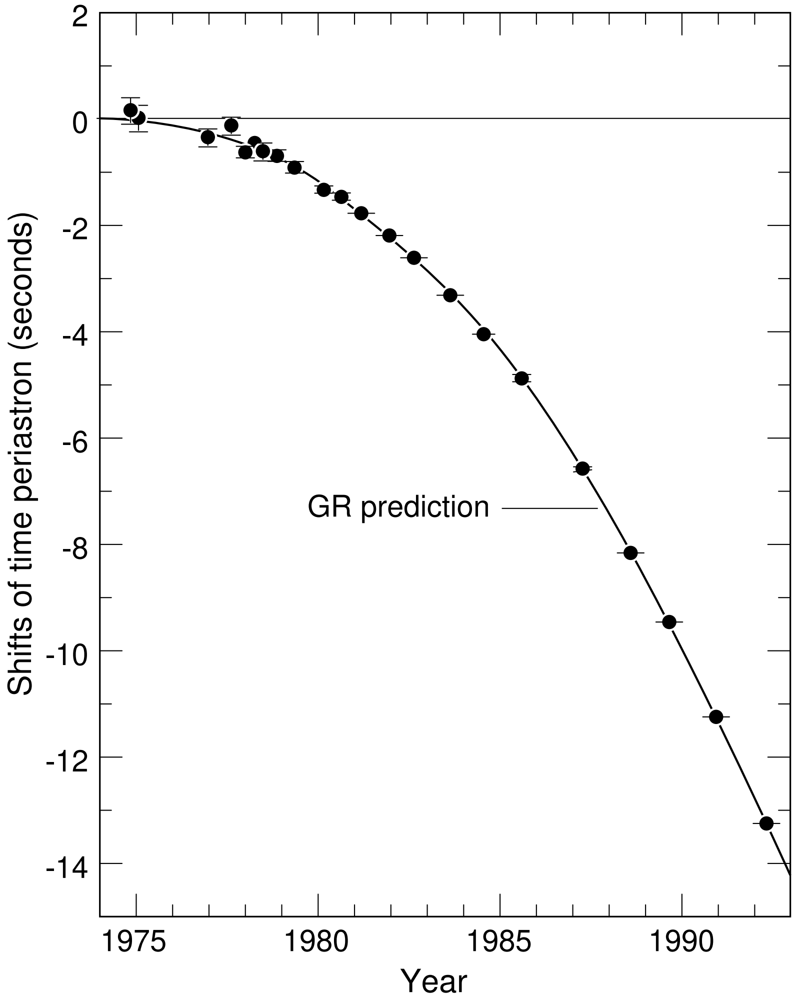

Figure 5: Accumulated shift of the time of periastron in the PSR1913+16 system relative to an assumed orbit with constant period. The curve shows the predictions of General Relativity for energy losses from gravitational radiation.