# Lecture Twenty: Theory of Relativity

## 20.1 The maximally symmetric 3–space

In three dimensions, the curvature tensor $R_{\alpha\beta\gamma\delta}$ has, in general, six independent components, each of which is a function of the coordinates. We therefore need to specify six functions to define the inner properties of a general three dimensional space. Clearly, the more symmetrical the space, the fewer the functions needed to specify its inner properties.

A *maximally symmetric space* is specified by just one number – the curvature $K$. Such spaces must clearly be *homogeneous and isotropic* since $K$ is independent of the coordinates.

The curvature tensor of a maximally symmetric space must take a particulary simple form. It must clearly depend on the constant $K$ and on the metric tensor $g_{\mu\nu}$. I will make the *ansatz* that the curvature tensor can be written as

$$R_{\lambda\rho\sigma\nu} = K\left(g_{\sigma\rho}g_{\lambda\nu} - g_{\nu\rho}g_{\lambda\sigma}\right). \tag{1a}$$

This is the simplest expression that satisfies the symmetry properties of $R_{\lambda\rho\sigma\nu}$ and contains just $K$ and the metric tensor. We will see later whether this assumption leads to a consistent model.

The Ricci tensor is given by

$$\begin{aligned} R_{\sigma\rho} &= g^{\lambda\nu}R_{\lambda\rho\nu\sigma} = -g^{\lambda\nu}R_{\lambda\rho\nu\sigma} \\ &= -K\left[g_{\sigma\rho}g_{\lambda\nu}g^{\lambda\nu} - g_{\nu\rho}g_{\lambda\sigma}g^{\lambda\nu}\right] \\ &= -K\left[Ng_{\sigma\rho} - g_{\nu\rho}\delta^{\nu}_{\sigma}\right] \\ &= -K(N-1)g_{\sigma\rho} \end{aligned} \tag{1b}$$

where $N$ is the number of spatial dimensions. The curvature scalar is

$$R = R^{\lambda}_{\lambda} = -N(N-1)K. \tag{1c}$$

As in our derivation of the Schwarzchild metric, the metric of an isotropic space must depend only on the rotational invariants

$$\mathbf{x}.d\mathbf{x}, \quad r^2 = \mathbf{x}^2, \quad d\mathbf{x}^2.$$

Thus the metric must take the form

$$\begin{aligned} ds^2 &= C(r)(\mathbf{x}.d\mathbf{x})^2 + D(r)d\mathbf{x}^2 \\ &= C(r)r^2dr^2 + D(r)\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]. \end{aligned}$$

Following our analysis of the Schwarzchild metric, we can simplify this by redefining the radial coordinate $r'^2 = r^2D(r)$. Dropping the primes, the metric can be written as

$$ds^2 = \beta(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2}$$

We have met this before – it is indentical to the space part of the Schwarzchild metric. We have shown that the non-vanishing affine terms are

$$\left.\begin{array}{rcl}
\Gamma^r_{rr} &=& \frac{1}{2\beta(r)}\frac{d\beta}{dr} \\
\Gamma^r_{\theta\theta} &=& -\frac{r}{\beta(r)} \\
\Gamma^r_{\phi\phi} &=& -\frac{r\sin^2\theta}{\beta} \\
\Gamma^\theta_{r\theta} &=& 1/r \\
\Gamma^\phi_{\phi r} &=& 1/r \\
\Gamma^\theta_{\phi\phi} &=& -\sin\theta\cos\theta \\
\Gamma^\phi_{\phi\theta} &=& \cos\theta/\sin\theta
\end{array}\right\}\ \text{Compare with the lecture notes on the Schwarzchild solution}$$

Writing the Ricci tensor in terms of the affine connections,

$$R_{\mu\kappa} = \frac{\partial\Gamma^\lambda_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\lambda\eta},$$

and after some algebra, we find that the non-zero components of $R_{\mu\kappa}$ are

$$R_{rr} = -\frac{1}{\beta r}\frac{d\beta}{dr}, \tag{3$a$}$$

$$R_{\theta\theta} = \left(\sin^2\theta\right)^{-1} R_{\phi\phi} = -\left(1 + \frac{r}{2\beta^2}\frac{d\beta}{dr} - \frac{1}{\beta}\right). \tag{3$b$}$$

But, according to our ansatz, equation (1b),

$$R_{\mu\nu} = -2Kg_{\mu\nu}, \tag{4}$$

and since the metric is given by equation (2), equations (3) and (4) require,

$$\frac{1}{\beta r}\frac{d\beta}{dr} = 2K\beta(r), \tag{5$a$}$$

$$1 + \frac{r}{2\beta^2}\frac{d\beta}{dr} - \frac{1}{\beta} = 2Kr^2. \tag{5$b$}$$

Integrating equation (5a),

$$\begin{array}{lrcl}
 & \displaystyle\int \frac{1}{\beta^2}\,d\beta &=& \displaystyle\int 2Kr\,dr, \\
\textit{i.e.,} & \displaystyle -\frac{1}{\beta} &=& Kr^2 + \text{constant}, \\
\text{and so,} & \beta(r) &=& \displaystyle\frac{1}{A - Kr^2}.
\end{array}$$

where $A$ is a constant. Now substitute this expression into equation (5b),

$$\begin{aligned}
1 + \frac{1}{2}\frac{r^2}{\beta}2K\beta - \frac{1}{\beta} &= 2Kr^2 \\
1 - A + Kr^2 &= Kr^2,
\end{aligned}$$

hence,

$$A = 1.$$

Thus, we have constructed a maximally symmetric 3–space with the metric

$$ds^2 = \frac{dr^2}{(1 - Kr^2)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{6}$$

which has a curvature tensor specified by the one number $K$ – the *curvature* of the space. We see that our guess for the form of the curvature tensor (1a) has led to a consistent form for the metric (6).

Notice also that this is *exactly* the same form as the metric for a three–sphere embedded in 4-dimensional Euclidean space that I discussed when first introducing Riemannian geometry. Notice also that the metric contains a "hidden symmetry". In this metric *the origin of the radial coordinate is completely arbitrary* – we can chose any point in this space as our origin since all points are equivalent. *There is no centre in this space.*

## 20.2 Simple Cosmological Models

We have derived a metric for maximally symmetric 3-space, but to describe a cosmological model, we need a metric for 4-dimensional space-time. I will write the metric of 4-dimensional space-time as

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{(1 - Kr^2)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]. \tag{7}$$

This is called the Friedmann–Robertson–Walker (FRW) metric, and describes the simplest cosmological models in which space is homogeneous and isotropic.

Why choose this form of metric? Firstly, when we look around us we see that the stars around us are grouped into a large density concentration – the Milky Way Galaxy. On a slightly larger scale, we see that our Galaxy belongs to a small group of galaxies (called the Local Group). Our Galaxy and our nearest large neighbour, the Andromeda galaxy, dominate the mass of the Local Group. On still larger scales we see that our Local Group sits on the outskirts of a giant supercluster of galaxies centred in the constellation of Virgo. Evidently on small scales matter is distributed in a highly irregular way, but as we look on larger and larger scales, the matter distribution looks more and more uniform. In fact, we have very good evidence (particularly from the constancy of the temperature of the cosmic microwave background in different directions on the sky) that the Universe is *isotropic* on the very largest scales to high accuracy. If the Universe has no preferred centre, then *isotropy* also implies *homogeneity.* There are therefore good physical reasons to study simple cosmological models in which the Universe is assumed to be homogeneous and isotropic.

If the matter distribution is homogeneous and isotropic, then space must be described by the maximally symmetric metric (6). The space must have a constant curvature $K$. However, we have not yet set the scale of the space – this is what the factor $R(t)$ does in equation (7). The factor $R(t)$ sets the 'scale' of the space and is often called the *scale factor*. It cannot depend on spatial coordinates, otherwise we would not be able to satisfy equations (3) and

(4), but there is no reason why it can't depend on the time coordinate $t$. You can also see that on scales small compared with the spatial curvature, the metric is equivalent to the Minkowski metric of Special Relativity.

To determine the function $R(t)$, we must solve the gravitational field equations in the presence of matter. Recall that the field equations are

$$\left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} \right) = -8\pi G T_{\mu\nu},$$

where I have included the cosmological constant $\Lambda$. As we have shown previously, an equivalent form of the field equations is

$$(R_{\mu\nu} - \Lambda g_{\mu\nu}) = -8\pi G S_{\mu\nu}, \tag{8a}$$

where the source term $S_{\mu\nu}$ is

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\mu}_{\mu}. \tag{8b}$$

Clearly, to solve these equations we need a model for the energy-momentum tensor of the matter that fills the Universe.

For simplicity, I will assume that the matter in the Universe is a perfect fluid. Such a fluid is characterised by a density $\rho$ and a pressure $p$ at every point, and the energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - \rho g_{\mu\nu}.$$

Since we are seeking solutions for a homogeneous and isotropic universe, the pressure and density $p$ and $\rho$ *must be functions of time $t$ alone.* The coordinates, $r$, $\theta$, and $\phi$ assigned to particles must remain fixed if the universe is to stay homogeneous, hence since

$$u^\mu = \frac{dx^\mu}{d\tau},$$

the components of the 4-velocity must be

$$u^0 = u_0 = 1, \qquad u^i = u_i = 0.$$

The FRW metric is diagonal with non-zero components

$$\begin{aligned}
g_{00} &= 1, \\
g_{rr} &= -\frac{R^2(t)}{(1 - Kr^2)} = -R^2 \bar{g}_{rr}, \\
g_{\theta\theta} &= -R^2(t) r^2 = -R^2 \bar{g}_{\theta\theta}, \\
g_{\phi\phi} &= -R^2(t) r^2 \sin^2\theta = -R^2 \bar{g}_{\phi\phi},
\end{aligned}$$

and its inverse has non-zero components

$$\begin{aligned}
g^{00} &= 1, \\
g^{rr} &= -\frac{(1 - Kr^2)}{R^2(t)} = -\frac{\bar{g}_{rr}}{R^2}, \\
g^{\theta\theta} &= -\frac{1}{R^2 r^2} = -\frac{\bar{g}_{\theta\theta}}{R^2}, \\
g^{\phi\phi} &= -\frac{1}{R^2 r^2 \sin^2\theta} = -\frac{\bar{g}_{\phi\phi}}{R^2(t)},
\end{aligned}$$

where I have written the metric of the constant curvature 3–space as (note the signature)

$$d\bar{s}^2 = \bar{g}_{ij} dx^i dx^j.$$

We can therefore write the FRW metric in terms of the metric coeffiecients of the 3–space as

$$ds^2 = dt^2 - R^2(t)\bar{g}_{ij} dx^i dx^j.$$

From our definition of the affine connection

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\kappa}\left(\frac{\partial g_{\gamma\kappa}}{\partial x^{\beta}} + \frac{\partial g_{\beta\kappa}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\kappa}}\right),$$

we can show straightforwardly that the components of the affine connection are

$$\begin{aligned}
\Gamma^{0}_{00} &= \frac{1}{2} g^{0\kappa}\left(\frac{\partial g_{0\kappa}}{\partial x^{0}} + \frac{\partial g_{0\kappa}}{\partial x^{0}} - \frac{\partial g_{00}}{\partial x^{\kappa}}\right) = 0, \\
\Gamma^{0}_{0i} &= \frac{1}{2} g^{00}\left(\frac{\partial g_{i0}}{\partial x^{0}} + \frac{\partial g_{00}}{\partial x^{i}} - \frac{\partial g_{0i}}{\partial x^{0}}\right) = 0, \\
\Gamma^{0}_{ij} &= \frac{1}{2} g^{00}\left(\frac{\partial g_{i0}}{\partial x^{j}} + \frac{\partial g_{j0}}{\partial x^{i}} - \frac{\partial g_{ij}}{\partial x^{0}}\right) = R(t)\dot{R}(t)\bar{g}_{ij}, \\
\Gamma^{i}_{00} &= \frac{1}{2} g^{i\kappa}\left(\frac{\partial g_{0\kappa}}{\partial x^{0}} + \frac{\partial g_{0\kappa}}{\partial x^{0}} - \frac{\partial g_{00}}{\partial x^{\kappa}}\right) = 0, \\
\Gamma^{i}_{j0} &= \frac{1}{2} g^{i\kappa}\left(\frac{\partial g_{0\kappa}}{\partial x^{j}} + \frac{\partial g_{j\kappa}}{\partial x^{0}} - \frac{\partial g_{j0}}{\partial x^{\kappa}}\right) \\
&= \frac{1}{2} g^{il}\frac{\partial g_{jl}}{\partial x^{0}} = \frac{1}{2R^2(t)} 2R\dot{R}\delta^{i}_{j} = \frac{\dot{R}(t)}{R(t)}\delta^{i}_{j}, \\
\Gamma^{i}_{jk} &= \frac{1}{2} g^{i\kappa}\left(\frac{\partial g_{j\kappa}}{\partial x^{k}} + \frac{\partial g_{k\kappa}}{\partial x^{j}} - \frac{\partial g_{jk}}{\partial x^{\kappa}}\right) \\
&= \frac{1}{2} g^{il}\left(\frac{\partial g_{jl}}{\partial x^{k}} + \frac{\partial g_{kl}}{\partial x^{j}} - \frac{\partial g_{jk}}{\partial x^{l}}\right) \\
&= \bar{\Gamma}^{i}_{jk},
\end{aligned}$$

where $\bar{\Gamma}^{i}_{jk}$ are the affine connection components of the constant curvature 3-space derived in the previous section.

Now that we have the components of the affine connection, we can readily compute the components of the Ricci tensor

$$\begin{aligned}
R_{00} &= \frac{\partial \Gamma^{\lambda}_{0\lambda}}{\partial x^{0}} - \frac{\partial \overbrace{\Gamma^{\lambda}_{00}}^{=0}}{\partial x^{\lambda}} + \Gamma^{\eta}_{0\lambda}\Gamma^{\lambda}_{0\eta} - \overbrace{\Gamma^{\eta}_{00}}^{=0}\Gamma^{\lambda}_{\lambda\eta} \\
&= \frac{\partial \Gamma^{i}_{0i}}{\partial x^{0}} + \Gamma^{i}_{0j}\Gamma^{j}_{0i} \\
&= \frac{\partial}{\partial t}\left(3\frac{\dot{R}(t)}{R}\right) + 3\left(\frac{\dot{R}}{R}\right)^{3} = 3\frac{\ddot{R}(t)}{R(t)},
\end{aligned}$$

$$
\begin{aligned}
R_{0i} &= \frac{\partial \Gamma^{\lambda}_{i\lambda}}{\partial x^0} - \frac{\partial \Gamma^{\lambda}_{0i}}{\partial x^{\lambda}} + \Gamma^{\eta}_{0\lambda}\Gamma^{\lambda}_{i\eta} - \Gamma^{\eta}_{0i}\Gamma^{\lambda}_{\lambda\eta} \\
&= \frac{\partial \Gamma^{0}_{i0}}{\partial x^0} + \frac{\partial \Gamma^{j}_{ij}}{\partial x^0} - \frac{\partial \Gamma^{0}_{0i}}{\partial x^0} - \frac{\partial \Gamma^{j}_{0i}}{\partial x^j} + \Gamma^{\eta}_{0j}\Gamma^{j}_{ln} - \Gamma^{j}_{0i}\Gamma^{\lambda}_{\lambda j} \\
&= \Gamma^{l}_{0j}\Gamma^{j}_{il} - \Gamma^{j}_{0i}\Gamma^{l}_{lj} = \frac{\dot{R}}{R}\delta^{l}_{j}\Gamma^{j}_{il} - \frac{\dot{R}}{R}\delta^{i}_{j}\Gamma^{l}_{lj} \\
&= \frac{\dot{R}}{R}\left(\Gamma^{l}_{li} - \Gamma^{l}_{li}\right) = 0, \\
R_{ij} &= \frac{\partial \Gamma^{\lambda}_{i\lambda}}{\partial x^j} - \frac{\partial \Gamma^{\lambda}_{ij}}{\partial x^{\lambda}} + \Gamma^{\eta}_{ij}\Gamma^{\lambda}_{j\eta} - \Gamma^{\eta}_{ij}\Gamma^{\lambda}_{\lambda\eta} \\
&= \frac{\partial \Gamma^{l}_{il}}{\partial x^j} - \frac{\partial \Gamma^{0}_{ij}}{\partial x^0} - \frac{\partial \Gamma^{l}_{ij}}{\partial x^l} + \Gamma^{0}_{i\lambda}\Gamma^{\lambda}_{j0} + \Gamma^{l}_{i\lambda}\Gamma^{\lambda}_{jl} - \Gamma^{0}_{ij}\Gamma^{\lambda}_{\lambda 0} - \Gamma^{l}_{ij}\Gamma^{\lambda}_{\lambda l} \\
&= -\frac{\partial \Gamma^{0}_{ij}}{\partial x^0} + \Gamma^{0}_{il}\Gamma^{l}_{j0} + \Gamma^{l}_{i0}\Gamma^{0}_{jl} - \Gamma^{0}_{ij}\Gamma^{l}_{l0} - \Gamma^{l}_{ij}\Gamma^{0}_{0l} + \underbrace{\bar{R}_{ij}}_{\downarrow\ \text{this is the Ricci tensor for the } \textit{spatial} \text{ 3-space}} \\
&= -\left(\dot{R}^2 + R\ddot{R}\right)\bar{g}_{ij} + R\dot{R}\bar{g}_{il}\frac{\dot{R}}{R}\delta^{l}_{j} + R\dot{R}\bar{g}_{jl}\frac{\dot{R}}{R}\delta^{l}_{i} - 3\frac{\dot{R}}{R}R\dot{R}\bar{g}_{ij} + \bar{R}_{ij} \\
&= \tilde{R}_{ij} - \left(2\dot{R}^2 + R\ddot{R}\right)\bar{g}_{ij}.
\end{aligned}
$$

But, remember that for a maximally symmetric 3–space

$$\bar{R}_{ij} = -2K\bar{g}_{ij}$$

hence,

$$R_{ij} = -2K\bar{g}_{ij} - \left(2\dot{R}^2 + R\ddot{R}\right)\bar{g}_{ij}.$$

Having found the components of the Ricci tensor, we now need to investigate the source term in the gravitational field equations,

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\mu}_{\mu}.$$

Starting from the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} - pg_{\mu\nu},$$

raise an index,

$$
\begin{aligned}
T^{\kappa}_{\nu} &= (\rho + p)u^{\kappa}u_{\nu} - p\delta^{\kappa}_{\nu}, \\
i.e. \qquad T^{\kappa}_{\kappa} &= (\rho + p) - 4p = \rho - 3p.
\end{aligned}
$$

So, the components of the source term are

$$\begin{aligned}
S_{00} &= (\rho + p) - p - \frac{1}{2}(\rho - 3p) \\
&= \frac{1}{2}(\rho + 3p), \\
S_{0i} &= 0, \\
S_{ij} &= pR^2(t)\bar{g}_{ij} + \frac{1}{2}(\rho - 3p)R^2(t)\bar{g}_{ij} \\
&= \frac{1}{2}(\rho - p)R^2(t)\bar{g}_{ij}.
\end{aligned}$$

## 20.3 Applying the field equations to Cosmology

Let us summarise what we have done so far. We seek a solution of the field equations

$$(R_{\mu\nu} - \Lambda g_{\mu\nu}) = -8\pi G S_{\mu\nu}.$$

The metric is given by equation (7), and we have computed the components of the Ricci tensor and the source term $S_{\mu\nu}$:

$$\left\{\begin{array}{lclclcl}
R_{00} &=& 3\ddot{R}/R, & \qquad & S_{00} &=& \frac{1}{2}(\rho + 3p), \\
R_{0i} &=& 0, & & S_{0i} &=& 0, \\
R_{ij} &=& -2K\bar{g}_{ij} - (2\dot{R}^2 + R\ddot{R})\bar{g}_{ij} & & S_{ij} &=& \frac{1}{2}(\rho - p)R^2\bar{g}_{ij}.
\end{array}\right\}$$

So all we have to do is to equate the two sides of the field equations. The field equation for $R_{0i}$ is satisfied trivially since all terms are zero. The equation for $R_{00}$ gives

$$3\frac{\ddot{R}}{R} = -4\pi G(\rho + 3p) + \Lambda, \tag{9a}$$

and the equation for $R_{ij}$ gives

$$2K + (2\dot{R}^2 + R\ddot{R}) = 4\pi G(\rho - p)R^2(t) + \Lambda R^2(t).$$

Eliminating $\ddot{R}$ from this equation

$$2K + 2\dot{R}^2 + R\left[-\frac{4\pi G}{3}(\rho + 3p)R + \Lambda\frac{R}{3}\right] = 4\pi G(\rho - p)R^2 + \Lambda R^2(t),$$

hence

$$2K + 2\dot{R}^2 = \frac{16\pi G\rho}{3}R^2 + \frac{2}{3}\Lambda R^2(t),$$

*i.e.*

$$\dot{R}^2 + K = \frac{8\pi G}{3}\rho R^2 + \frac{1}{3}\Lambda R^2. \tag{9b}$$

Equations (9a) and (9b) give us two differential equations describing the time evolution of the scale factor $R(t)$. They are derived from the gravitational field equations and so the solutions of equations (9) will yield cosmological models that are fully consistent with General Relativity.

There is one further important equation that we can derive. Energy conservation requires that the covariant derivative of the energy momentum tensor is zero

$$T^{\mu\nu}_{\ ;\nu} = 0.$$

The components of $T^{\mu\nu}$ are

$$\left.\begin{array}{rcl} T^{\mu\nu} &=& (\rho + p)u^\mu u^\nu - pg^{\mu\nu}, \\ T^{00} &=& \rho, \\ T^{0i} &=& 0, \\ T^{ij} &=& -pg^{ij} = (p/R^2)\bar{g}^{ij}, \end{array}\right\}$$

and the covariant derivative is

$$T^{\mu\nu}_{\ ;\lambda} = \frac{\partial T^{\mu\nu}}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa} T^{\kappa\nu} + \Gamma^\nu_{\lambda\kappa} T^{\kappa\mu},$$

and so

$$T^{\mu\nu}_{\ ;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu\kappa} T^{\kappa\nu} + \Gamma^\nu_{\nu\kappa} T^{\kappa\mu}.$$

In fact, if you work through the algebra you will find that $T^{i\nu}_{\ ;\nu}$ is satisfied trivially, but we get an important equation from the requirement that $T^{0\nu}_{\ ;\nu} = 0$:

$$\begin{aligned} T^{0\nu}_{\ ;\nu} &= \frac{\partial T^{00}}{\partial x^0} + \Gamma^0_{\nu\kappa} T^{\kappa\nu} + \Gamma^\nu_{\nu\kappa} T^{\kappa 0} \\ &= \frac{\partial T^{00}}{\partial x^0} + \Gamma^0_{ij} T^{ij} + \Gamma^j_{j0} T^{00}, \end{aligned}$$

*i.e.*

$$\frac{\partial \rho}{\partial t} + \frac{\dot{R}}{R}\bar{g}_{ij} p \bar{g}^{ij} + 3\frac{\dot{R}}{R}\rho = 0,$$

which we can write compactly as

$$\frac{1}{R^3}\frac{d(\rho R^3)}{dt} = -3p\frac{\dot{R}}{R},$$

or equivalently as

$$\frac{d(\rho R^3)}{dR} = 3pR^2. \tag{9c}$$

This last equation expresses energy conservation.

## 20.4 Summary: The Friedmann equations

For a homogeneous, isotropic universe composed of a perfect ideal fluid of density $\rho$ and pressure $p$, the scale factor $R(t)$ of the FRW metric must satisfy the equations:

$$\begin{aligned} \frac{3\ddot{R}}{R} &= -4\pi G(\rho + 3p) + \Lambda, \\ \dot{R}^2 + K &= \frac{8\pi G}{3}\rho R^2 + \frac{1}{3}\Lambda R^2. \end{aligned}$$

These equations are often called the *Friedmann equations*. (When the cosmological constant is included these equations are sometimes called the Friedmann-Lemaitre equations).

Energy conservation relates the density, pressure and scale factor via the equation

$$\frac{d}{dR}(\rho R^3) = 3pR^2.$$

We will investigate solutions of these equations in the next Section.