Lecture Two: Theory of Relativity

2.1 Elementary (but profound) consequences of Lorentz the transformation

2.1.1 Length Contraction

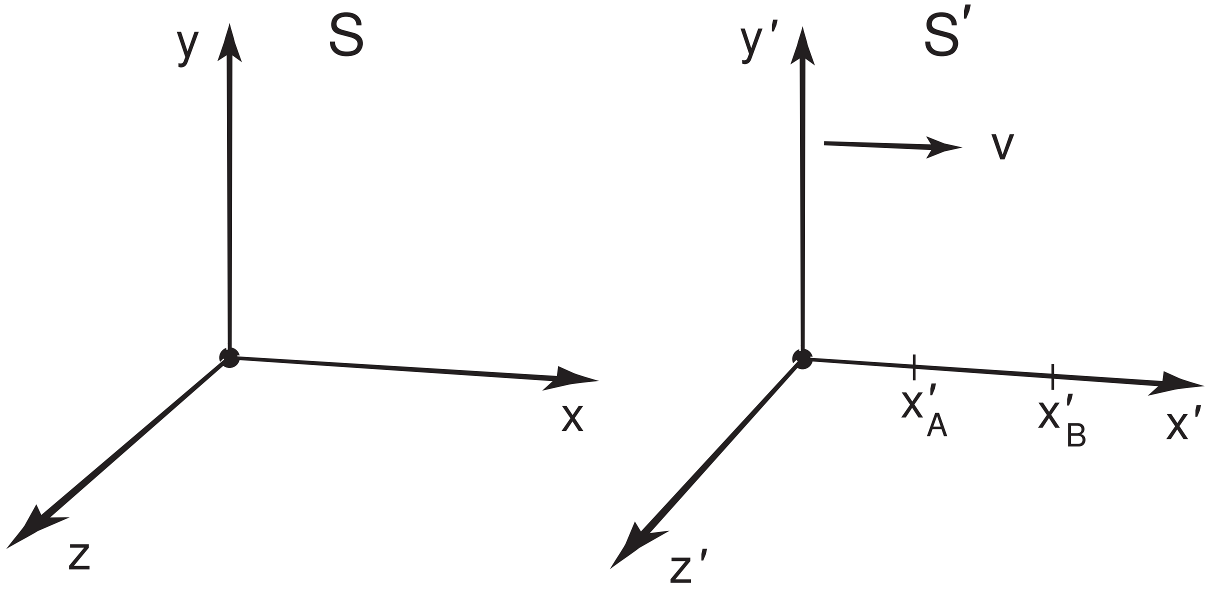

Figure 1:

Consider a rod in frame S' of proper length

$$\ell_0 = x'_A - x'_B \quad .$$

We want to apply the Lorentz transformation formulae and see what length an observer in frame S assigns to the rod. Applying the Lorentz transformation,

$$\begin{aligned} x'_A &= \gamma \left(x_A - v t_A / c^2 \right), \\ x'_B &= \gamma \left(x_B - v t_B / c^2 \right), \end{aligned}$$

relates the coordinates of the ends of the rod in S' to the coordinates in S. The observer in S measures the length of the rod at a *fixed* time $t = t_A = t_B$ as

$$\ell = x_B - x_A = \frac{1}{\gamma} \left(x'_B - x'_A \right) = \frac{\ell_0}{\gamma}.$$

Hence the rod in S appears contracted by an amount

$$\ell = \ell_0 \left(1 - v^2/c^2 \right)^{1/2} .$$

2.1.2 Time Dilation

Suppose we have a clock at x'_A in S' and we record two successive events separated by time T_0.

The times recorded in S are:

$$\begin{aligned} t_1 &= \gamma\,(t'_1 - v x'_A/c^2)\,, \\ t_2 &= \gamma\,(t'_1 + T_0 - v x'_A/c^2)\,. \end{aligned}$$

Subtracting

$$T = t_2 - t_1 = \gamma T_0 = \frac{T_0}{(1 - v^2/c^2)^{1/2}},$$

hence, the moving clock goes slower by a factor of $(1 - v^2/c^2)^{1/2}$ (time dilation).

Note that an **ideal clock** is one that is unaffected by acceleration – external forces act identically on all parts of the clock (example is a muon).

For an ideal clock, its **instantaneous rate** depends on its **instantaneous speed** .

2.2 Space-time approach to special relativity (4-tensors)

Poincaré transformations leave invariant the squared interval between neighbouring events

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

This is analogous to the invariance of the squared interval

$$dx^2 + dy^2 + dz^2$$

under rotations and translations of orthogonal Cartesian coordinates in Euclidean space.

Four tensor

Is an object defined on the 4-space of events

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

which behaves as a **tensor** under a Poincaré transformation.

But what is a tensor?

• Construct a set of real variables $\{x^1,\ x^2,\ \ldots x^N\}$ which we can regard as coordinates of an N-dimensional space V_N.

• A non-singular transformation of these coordinates to a new set of coordinates is a reco-ordination of the space V_N.

• A tensor is an object in V_N described by a set of M real numbers.

• The components of the tensor (*i.e.* the M numbers) will (generally) change when we transform the coordinates of V_N.

• To qualify as a tensor, **the components must transform in a specific way**.

In N-dimensional space, a tensor of rank r has

$$\begin{aligned}
& M = N^r && \text{components} \\
r = 0, \quad & M = 1, && \text{defining a scalar} \\
r = 1, \quad & M = N, && \text{defining a vector} \\
r = 2, \quad & M = N^2 && \text{defining a second rank tensor}
\end{aligned}$$

and so on. So, to understand what a tensor is, we must investigate coordinate transformations.

Coordinate transformations

The space V_N is characterised by coordinates:

$$\begin{aligned}
& (x^1,\ x^2,\ x^3\ \ldots x^N) && \equiv (i,\ j,\ k,\ \ldots) \\
& (x'^1,\ x'^2,\ x'^3,\ \ldots x'^N) && \equiv (i',\ j',\ k',\ \ldots) \\
& \text{etc.}
\end{aligned}$$

Assume that the transformations from one set of coordinates to another are non-singular and differentiable

$$\frac{\partial x^{i'}}{\partial x^i} = p_i^{i'}, \quad \frac{\partial x^i}{\partial x^{i'}} = p_{i'}^i, \quad etc.$$

Definition of Tensors

- **Contravariant tensor**

Has components: $A^{i,j,...n}$ in x^i system

$A^{i',j',...n'}$ in $x^{i'}$ system

transforms according to $A^{i',j',...n'} = A^{i,j,...n}\, p^{i'}_i p^{j'}_j \ldots p^{n'}_n$

- **Covariant tensor**

Has components: $A_{i,j,...n}$ in x^i system

$A_{i',j',...n'}$ in $x^{i'}$ system

transforms according to $A_{i',j',...n'} = A_{i,j,...n}\, p^i_{i'}, p^j_{j'} \ldots p^n_{n'}$

- **Mixed tensor**

Has components: $A^{i...k}_{l...n}$ in x^i system

transforms according to $A^{i'...k'}_{\ell'...n'} = A^{i...k}_{\ell...n}\, p^{i'}_i \ldots p^{k'}_k p^{\ell}_{\ell'} \ldots p^n_{n'}$.

Note the following points:

- The components of tensors in the new coordinate system are linear functions of the components in the old coordinate system.

- An object is a tensor if it transforms as a tensor under all non-singular coordinate transformations.

- We can construct objects which behave as tensors only under certain *subgroups* of linear transformations, *e.g.* Lorentz transformations.

2.3 Examples of tensors

- **Contravariant vector**

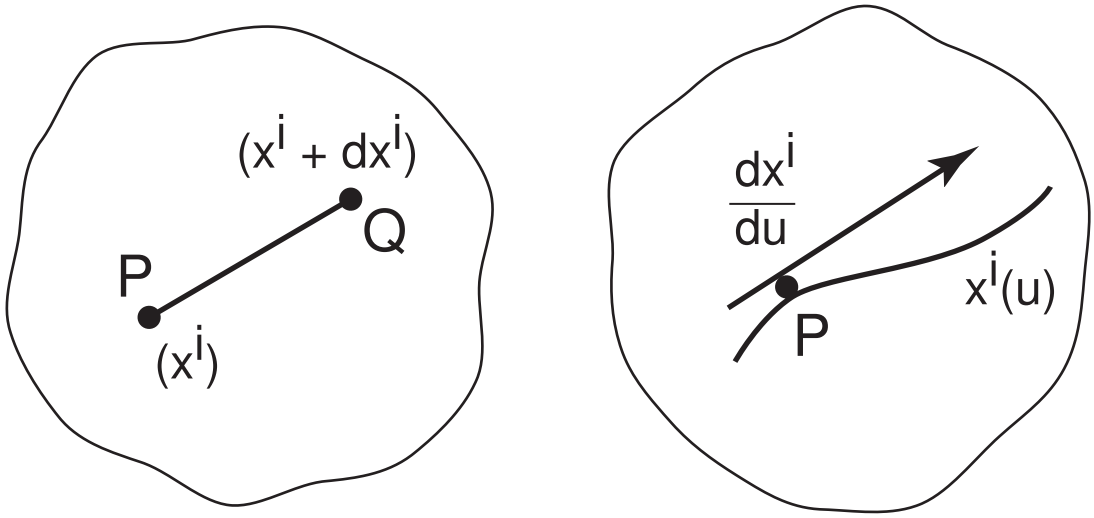

Figure 2:

Consider a curve $x^i(u)$ (u is a parameter that varies along the curve) the tangent vector dx^i/du at some point P is a contravariant vector.

The derivatives of coordinates transform as a contravariant vector since

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i.$$

- **Covariant Vector**

The gradient of a scalar field $\phi(x^1, \ \ldots, x^n)$ is an example of a covariant vector since the gradient transforms as

$$\frac{\partial \phi}{\partial x^{i'}} = \frac{\partial \phi}{\partial x^i} \ \frac{\partial x^i}{\partial x^{i'}}.$$

2.4 Why use tensors?

Suppose that we have two tensors,

$$X_{ij} = Y_{ij}.$$

Apply a transformation of the coordinates,

$$\begin{aligned} X_{i'j'} &= X_{ij} \ \tfrac{\partial x^i}{\partial x^{i'}} \ \tfrac{\partial x^j}{\partial x^{j'}}, \\ Y_{i'j'} &= Y_{ij} \ \tfrac{\partial x^i}{\partial x^{i'}} \ \tfrac{\partial x^j}{\partial x^{j'}}, \end{aligned}$$

and so,

$$X_{i'j'} = Y_{i'j'}.$$

i.e. a **tensor equation** is **invariant** under all non-singular coordinate transformations and is therefore **coordinate free**.

We would like to formulate the laws of physics so that they are manifestly independent of the coordinate system. The above argument suggests that the laws of physics are formulated as tensorial equations.

2.5 Space-time

The space of events in special relativity is endowed with a metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where $g_{\mu\nu}$ is constant under Lorentz transformations

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

The metric tensor $g_{\mu\nu}$ defines the scalar product of two four vectors,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= g_{\mu\nu} A^\mu B^\mu \\ &= A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3. \end{aligned}$$

and (given our definition of tensors) is invariant under any coordinate transformation.

The metric tensor $g_{\mu\nu}$ can be used to raise and lower indices, since we can write our scalar product

$$\mathbf{A} \cdot \mathbf{B} = g_{\mu\nu} A^\mu B^\nu,$$

as

$$\mathbf{A} \cdot \mathbf{B} = A_\nu B^\nu,$$

where

$$A_\nu = g_{\mu\nu} A^\mu$$

is the covariant form of the vector A^μ. So, $g_{\mu\nu}$ acts to *contract* the index μ of A^μ.

Now, we can define a tensor

$$g^{\mu\nu}$$

that will *raise* the index of a vector:

$$\begin{aligned} A^\mu &= g^{\mu\nu} A_\nu \\ A^\mu &= g^{\mu\nu}[g_{\lambda\nu} A^\lambda] \\ &= g^{\mu\nu} g_{\lambda\nu} A^\lambda. \end{aligned}$$

Evidently $g^{\mu\nu}$ must satisfy

$$g^{\mu\nu} g_{\lambda\nu} = \delta^\mu_\lambda,$$

where δ^μ_λ in the Kronecker δ,

$$\delta^\mu_\lambda = \begin{cases} 1 & \mu = \lambda, \\ 0 & \mu \neq \lambda, \end{cases}$$

i.e. $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. Now in Special Relativity the metric tensor $g_{\mu\nu}$ is diagonal and equal to

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

and so the inverse $g^{\mu\nu}$ is identical,

$$(g^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

We can use the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ to raise and lower indices of tensors of arbitrary rank. For example, we can produce a mixed tensor from the second rank tensor $B^{\mu\lambda}$ as follows

$$B^\mu_\nu = g_{\nu\lambda} B^{\mu\lambda}.$$

Now, returning to the discussion of space-time, consider an event P at (x^α) and all events in its neighbourhood $(x^\alpha + \Delta x^\alpha)$ as shown in Figure 3 below.

We can group the neighbouring events into classes according to whether the (invariant) metric interval

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

is positive, zero or negative.

• $\Delta s^2 = 0$ - defines a light cone, *i.e.* a spherical wavefront converging on P. Events lying on the light-cone are termed 'light-like' or 'null'.

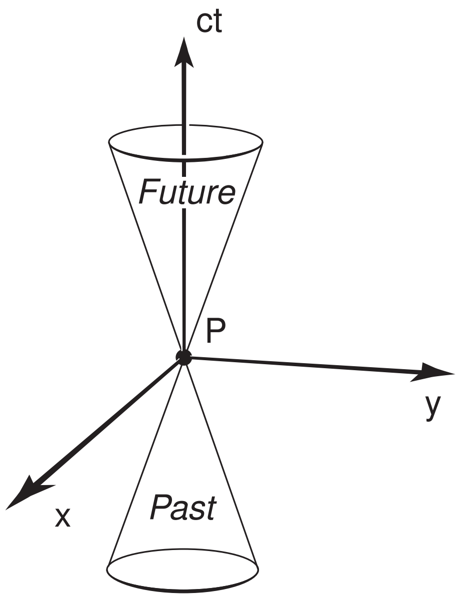

Figure 3:

- $\Delta s^2 > 0$ - events with $\Delta s^2 > 0$ lie inside the light-cone and are termed 'time-like'.
- $\Delta s^2 < 0$ - events with $\Delta s^2 < 0$ lie outside light-cone and are termed 'space-like'.

What is the significance of these classifications? This is made clear in the following diagram:

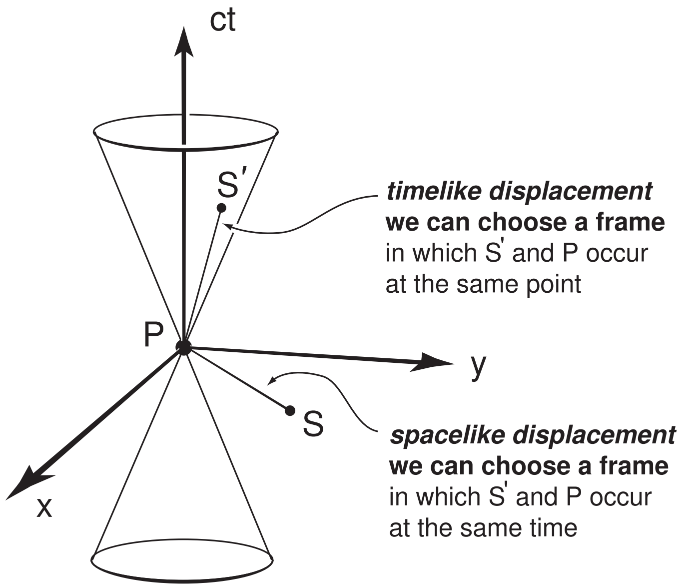

Figure 4:

This shows two events S' and S. S' lies within the light cone of P and so is displaced by a timelike interval. If the interval is timelike, we can find an inertial frame in which the events S' and P appear at the same point in space.

The events P and S are separated by a spacelike interval. For these we can choose an inertial frame in which the events S and P appear to occur at the same time.

These points become obvious if we view Lorentz transformations as rotations, as will be discussed in the lectures.

2.6 Four-velocities and the velocity addition law

The proper time interval between two events is

$$d\tau^2 = \frac{ds^2}{c^2} = dt^2 - \frac{(dx^2 + dy^2 + dz^2)}{c^2}.$$

For an ideal clock at rest

$$d\tau^2 = dt^2,$$

For an ideal clock moving at speed u with respect to an observer,

$$\begin{aligned} d\tau^2 &= dt^2 - u^2/c^2 dt^2, \\ &= dt^2 \left(1 - u^2/c^2\right), \end{aligned}$$

so

$$dt = d\tau/(1 - u^2/c^2), \qquad \text{(time dilation.)}$$

Consider a particle with world line $x^\mu(\tau)$, then

$$\begin{array}{ll} \dfrac{dx^\mu}{d\tau} \quad \text{is a four} - \text{vector :} & \text{the four} - \text{velocity } \mathbf{U} \\ \dfrac{d^2x^\mu}{d\tau^2} \quad \text{is also a four} - \text{vector :} & \text{the four} - \text{acceleration } \mathbf{A} \end{array}$$

The four-velocity of the particle is therefore

$$\mathbf{U} = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = (\gamma c,\ \gamma \mathbf{u}),$$

where $\mathbf{u}$ is

$$\mathbf{u} = \frac{d\mathbf{x}}{dt}.$$

Now recall the Lorentz transformation of the coordinates between two inertial frames,

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

The four-velocity **U** is a four-vector, and so according to the transformation rules for tensors will transform just like the components of the coordinates. The transformation formulae for the four-velocity will tell us how velocities transform between inertial frames *i.e.* the *velocity addition law* in Special Relativity. Inserting the components of the four-velocity, $(\gamma c, \gamma \mathbf{u})$ into the Lorentz transformation equations for the coordinates gives

$$\begin{aligned}
\gamma(u') &= \gamma(v)\left[\gamma(u) - v\gamma(u)u_1/c^2\right], \\
\gamma(u')u_1' &= \gamma(v)\left[\gamma(u)u_1 - v\gamma(u)\right], \\
\gamma(u')u_2' &= \gamma(u)u_2, \\
\gamma(u')u_3' &= \gamma(u)u_3.
\end{aligned}$$

From the first of these equations

$$\frac{\gamma(u)}{\gamma(u')} = \frac{1}{\gamma(v)}\,\frac{1}{(1-u_1 v/c^2)}.$$

And so the velocity addition law in Special Relativity is:

$$\begin{aligned}
u_1' &= \frac{(u_1 - v)}{(1-u_1 v/c^2)}, \\
u_2' &= \frac{u_2}{\gamma(v)\,(1-u_1 v/c^2)}, \\
u_3' &= \frac{u_3}{\gamma(v)\,(1-u_1 v/c^2)}.
\end{aligned}$$

Note that the u_2' and u_3' components are affected, even though the relative motion of the two frames is in the x (component 1) direction. This is a result of time dilation.

Notice that in this example, the tensorial transformation rules allowed us to derive the velocity addition law in amost trivial way by inspection of the Lorentz transformation formulae for the coordinates.