Lecture Six: Theory of Relativity

6.1 Freely falling particles

In this section, we will study the motion of a particle that moves purely under the action of gravity (a *freely falling* particle). According to the principle of equivalence, we can construct a local inertial coordinate system ξ^μ in which the particle moves in a straight line.

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{1}$$

and, for small displacements, $d\xi^\mu$, $d\xi^\nu$, the metric is the ordinary metric of special relativity

$$ds^2 = c^2 d\tau^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu,$$

$$(\eta_{\mu\nu} = +1,\ -1,\ -1,\ -1).$$

Now, starting from equation (1), transform to a general coordinate system x^μ,

$$\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{2}$$

Multiply (2) by

$$\frac{\partial x^\lambda}{\partial \xi^\alpha}$$

and use the product rule

$$\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta^\lambda_\mu,$$

equation (2) is

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{3}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\ \frac{\partial^2\xi^\alpha}{\partial x^\mu dx^\nu} \tag{4}$$

this is called the affine connection

6.2 Relationship of $\Gamma^\lambda_{\mu\nu}$ to $g_{\mu\nu}$

Equation (3) isn't very useful to us yet, because we haven't yet specified how to calculate the *affine connection* $\Gamma^\lambda_{\mu\nu}$ at every point in space. In fact, the affine connection is related to the metric tensor. The invariance of the interval ds^2 means that the locally Euclidean coordinates ξ^μ are related to general coordinates x^μ according to

$$g_{\mu\nu} = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}. \tag{5}$$

Differentiate $g_{\mu\nu}$ with respect to x^λ,

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2\xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\lambda \partial x^\nu}\eta_{\alpha\beta},$$

and use equation (4) to eliminate the second derivatives of ξ

$$
\begin{aligned}
\frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= \Gamma^\rho_{\mu\nu}\frac{\partial \xi^\alpha}{\partial x^\rho}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \Gamma^\rho_{\lambda\nu}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x\rho}\eta_{\alpha\beta} \\
&= \Gamma^\rho_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\rho\mu},
\end{aligned}
$$

where the last line follows from equation (5).

Now, permute the indices and form the combination,

$$
\begin{aligned}
\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} &= g_{\rho\nu}\Gamma^\rho_{\lambda\mu} + g_{\rho\mu}\Gamma^\rho_{\lambda\nu} \\
&+ g_{\rho\nu}\Gamma^\rho_{\mu\lambda} + g_{\rho\lambda}\Gamma^\rho_{\mu\nu} \\
&- g_{\rho\lambda}\Gamma^\rho_{\nu\mu} - g_{\rho\mu}\Gamma^\rho_{\nu\lambda}. \qquad (6)
\end{aligned}
$$

I have arranged this equation in the form above so that you can see clearly what happens next. From equation (4), you can see that $\Gamma^\rho_{\nu\mu}$ is symmetric with respect to the indices ν and μ, so four of the terms on the right hand side of (6) cancel giving

$$
2 g_{\rho\nu}\Gamma^\rho_{\mu\lambda} = \left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right].
$$

Multiply this equation by $g^{\sigma\nu}$ and recall that

$$
g^{\sigma\nu} g_{\rho\nu} = \delta^\sigma_\rho
$$

we get,

$$
\Gamma^\sigma_{\mu\lambda} = \frac{1}{2} g^{\sigma\nu}\left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right]. \qquad (7)
$$

This equation is sometimes written as,

$$
\Gamma^\sigma_{\mu\lambda} = \left\{ \begin{matrix} \sigma \\ \mu\lambda \end{matrix} \right\},
$$

where the term in curly brackets (called a 'Christoffel symbol') represents the right hand side of equation (7). You will see Christoffel symbols in some of the older books on General Relativity, but I will not use this arcane notation in these lectures.

Equation (7) is very important, because it tells us how to compute the affine connections at any point in space. If I know the metric ($g_{\mu\nu}$) of the space, in some coordinate system x^μ, then I can form the derivatives of $g_{\mu\nu}$ appearing in (7) and hence calculate *all* the components of $\Gamma^\sigma_{\mu\lambda}$ at *any* point.

6.3 Transformation of the affine connection

You may have wondered why I have called the quantity $\Gamma^\lambda_{\mu\nu}$ the 'affine connection' rather than say the 'connection tensor', or some other sort of tensor. Let me first explain why I am not calling this quantity a tensor (we will deal with the terms 'affine' and 'connection' later). Recall the definition of the affine connection,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}. \tag{8}$$

Let's calculate how $\Gamma^{\lambda}_{\mu\nu}$ transforms when we change coordinates from $x^{\mu} \to x'^{\mu}$:

$$\begin{aligned}
\Gamma'^{\lambda}_{\mu\nu} &= \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\sigma}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right) \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \left[\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\sigma} \partial x^{\tau}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\nu} \partial x'^{\mu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right].
\end{aligned}$$

But, from our definition (8)

$$\Gamma'^{\lambda}_{\mu\nu} = \underbrace{\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \Gamma^{\rho}_{\sigma\tau}}_{\substack{\uparrow \\ \text{This transforms as we would} \\ \text{expect for a mixed tensor}}} + \underbrace{\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\nu} \partial x'^{\mu}}}_{\substack{\uparrow \\ \text{but this does not!}}}$$

If the two coordinate systems x and x' are related to each other by a *non-linear* transformation, the second derivative on the right hand side may be non-zero and so $\Gamma^{\lambda}_{\mu\nu}$ will not obey the transformation rules for a mixed tensor. $\Gamma^{\lambda}_{\mu\nu}$ is therefore not a tensor under *general* coordinate transformations (it behaves as a tensor only for *linear* transformations). Beware of objects with indices – they may not be tensors. The indices may simply be convenient labels rather than telling you how the components transform. It is the transformation rules, not the indices, that define whether an object is a tensor, and according to the transformation rules the affine connection is not, in general, a tensor. (Incidently, this is why some of the older books use the curly bracket 'Christoffel symbol' notation, so that the non-tensorial affine terms are clearly distinguishable from tensors. The practice has died out because their is no real possibility of confusion if you remember that $\Gamma^{\lambda}_{\mu\nu}$ is not a tensor.)

6.4 Freely falling particles again

But if $\Gamma^{\sigma}_{\mu\nu}$ is not a tensor, then what about the covariance of the equation describing the motion of a freely falling particle,

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} = 0? \tag{9}$$

First, lets check that equation (9) gives a sensible result in the limit of Special Relativity. In this limit,

$$\Gamma^{\mu}_{\nu\lambda} = 0, \qquad \frac{d^2 x^{\mu}}{d\tau^2} = 0.$$

The second of these equations tells us that the particle moves in a straight line, so equation (9) has the correct limiting behaviour. Now, lets look at how equation (9) transforms under the coordinate transformation $x^\mu \to x'^\mu$,

$$\frac{d^2 x'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial x'^\mu}{\partial x^\nu}\frac{d^2 x^\nu}{dt^2} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau} \tag{10}$$

and we have already derived the transformation law for $\Gamma^\mu_{\nu\lambda}$

$$\Gamma'^\mu_{\nu\lambda} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\kappa}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\kappa\sigma} + \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\lambda \partial x'^\nu}$$

so multiply this $\frac{dx'^\nu}{d\tau}\frac{dx'^\lambda}{d\tau}$

$$\Gamma'^\mu_{\lambda\nu}\frac{dx'^\mu}{d\tau}\frac{dx'^\lambda}{d\tau} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{dx^\kappa}{d\tau}\frac{dx^\sigma}{d\tau}\Gamma^\rho_{\kappa\sigma} + \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\lambda \partial x'^\nu}\frac{dx'^\nu}{d\tau}\frac{dx'^\lambda}{d\tau}. \tag{11}$$

But, the product rule is

$$\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu} = \delta^\lambda_\nu$$

so, differentiating with respect to x'^μ

$$\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\mu} = 0$$

and so

$$\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} = -\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\mu}.$$

Using this relation in equation (11)

$$\begin{aligned}
\Gamma'^\mu_{\lambda\nu} dx'^\nu d\tau \frac{dx'^\lambda}{d\tau} &= \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x \kappa}{\partial \tau}\frac{\partial x^\sigma}{\partial \tau}\Gamma^\rho_{\kappa\sigma} - \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\alpha}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x'^\nu}{d\tau}\frac{\partial x'^\lambda}{d\tau} \\
&= \frac{\partial x'^\mu}{\partial x^\rho}\frac{dx^\kappa}{d\tau}\frac{dx^\sigma}{d\tau}\Gamma^\rho_{\kappa\sigma} - \underbrace{\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\alpha}\frac{dx^\rho}{d\tau}\frac{dx^\alpha}{d\tau}}_{\text{but this last term \textbf{cancels} the last term in equation (10).}}
\end{aligned}$$

The equation of motion therefore transforms as

$$\underbrace{\frac{d^2 x'^\mu}{d\tau} + \Gamma'^\mu_{\nu\lambda}\frac{dx'^\nu}{d\tau}\frac{dx'^\lambda}{d\tau}}_{\text{so this transforms as a contravariant vector, and the sum of the two terms is a tensor.}} = \frac{\partial x'^\mu}{\partial x^\nu}\left[\frac{\partial^2 x^\nu}{\partial \tau^2} + \Gamma^\rho_{\kappa\sigma}\frac{dx^\kappa}{d\tau}\frac{dx^\sigma}{d\tau}\right]$$

The equation of motion of a freely falling particle is therefore a tensor equation and is generally covariant.

6.5 Motion of a particle in a weak gravitational field

Lets start with the equation of motion

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\nu}\frac{dx^2}{d\tau}\frac{dx^\lambda}{d\tau} = 0. \tag{12}$$

and let us assume that the particles moves slowly

$$\frac{d\mathbf{x}}{d\tau} \ll \frac{dt}{d\tau}, \quad v \ll c.$$

If the particle moves slowly, *we can ignore the three-velocity terms* in (12),

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\, c^2\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{13}$$

Assume further that the gravitational field is stationary, then all time derivatives of $g_{\mu\nu}$ are zero.

Recall the equation relating the affine connection to the metric tensor,

$$\Gamma^\mu_{\lambda\nu} = \frac{1}{2}g^{\kappa\mu}\left\{\frac{\partial g_{\nu\kappa}}{\partial x^\lambda} + \frac{\partial g_{\lambda\kappa}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\kappa}\right\}.$$

Then the components appearing in equation (13) are

$$\begin{aligned}\Gamma^\mu_{00} &= \frac{1}{2}g^{\kappa\mu}\left\{\frac{\partial g_{0\kappa}}{\partial x^0} + \frac{\partial g_{0\kappa}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\kappa}\right\}\\ &= -\frac{1}{2}g^{\kappa\mu}\frac{\partial g_{00}}{\partial x^\kappa}.\end{aligned}$$

Since we are assuming that the gravitational field is weak, the metric must be *nearly* Minkowski:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} \ll \eta_{\mu\nu},$$

and so, to first order in $h_{\mu\nu}$

$$\begin{aligned}\Gamma^\mu_{00} &= -\frac{1}{2}\eta^{\kappa\mu}\frac{\partial h_{00}}{\partial x^\kappa},\\ \Gamma^0_{00} &= -\frac{1}{2}\frac{\partial h_{00}}{\partial x^0} = 0,\\ \Gamma^i_{00} &= \frac{1}{2}\frac{\partial h_{00}}{\partial x^i},\end{aligned}$$

where the latin index in the last line runs over the spatial dimensions ($i = 1, 2, 3$).

Inserting these components into the equation of motion (13) gives

$$\begin{aligned}\frac{d^2t}{d\tau^2} &= 0, \qquad \text{i.e. } \frac{dt}{d\tau} = \text{constant},\\ \frac{d^2\mathbf{x}}{d\tau^2} &= -c^2\left(\frac{dt}{d\tau}\right)^2\frac{1}{2}\boldsymbol{\nabla}h_{00}.\end{aligned}$$

But, since $dt/d\tau$ = constant, we can combine these two equations to give the following equation of motion

$$\frac{d^2x}{dt^2} = -\frac{1}{2}c^2\boldsymbol{\nabla}h_{\circ\circ}.$$

Now, compare this equation with the usual *Newtonian* equation of motion of a particle in a gravitational field,

$$\frac{d^2\mathbf{x}}{dt^2} = -\boldsymbol{\nabla}\phi$$

they are *identical*, if we set

$$h_{00} = \frac{2\phi}{c^2}.$$

Hence our theory tends to the Newtonian limit if the metric in a weak gravitational field has

$$g_{00} = \left(1 + \frac{2\phi}{c^2}\right).$$

How big is the correction to the Minkowski metric? Here are some values of ϕ/c^2 for various systems:

$$\phi/c^2 = \begin{cases} 10^{-9} & \text{at the surface of the Earth} \\ 10^{-6} & \text{at the surface of the Sun} \\ 10^{-5} & \text{in the early Universe (cosmic microwave background)} \\ 10^{-4} & \text{at the surface of a white dwarf.} \end{cases}$$

You see that even at the surface of a dense object like a white dwarf, the value of ϕ/c^2 is much smaller than unity and hence the weak field limit will be an excellent approximation.

6.6 Time dilation in a gravitational field

For a standard clock at rest in gravitational field ϕ,

$$\begin{aligned} c^2\Delta\tau^2 &= g_{\mu\nu}dx^\mu dx^\nu, \\ &= g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}dt^2, \\ &\approx g_{00}\,c^2dt^2, \end{aligned}$$

so

$$dt = \frac{\Delta\tau}{(1+2\phi/c)^{1/2}}.$$

This equation tells us that a standard clock will go slow in a gravitational field. Likewise, the frequency of light will be affected, so at two points in the field,

$$\frac{\nu_2}{\nu_1} = \frac{(1+2\phi(x_2)/c^2)^{1/2}}{(1+2\phi(x_1)/c^2)^{1/2}}.$$

If ϕ/c^2 is assumed small, we can expand the terms in brackets

$$
\begin{aligned}
1 + \Delta\nu/\nu &= \{1 + \phi(x_2)/c^2 + \ldots\}\{1 - \phi(x_1)/c^2 + \ldots\} \\
\Delta\nu/\nu &= (\phi(x_2) - \phi(x_1))/c^2
\end{aligned}
$$

This formula is the basis of the famous Pound and Rebka experiment. The authors used the Mossbauer effect to measure the change in frequency of X-rays as they propagated down a tower of height 22.6m. According to our equation, the change in the gravitational potential over this distance should lead to a tiny change in frequency of $\Delta\nu/\nu = 2.46 \times 10^{-15}$. Pound and Rebka measured $\Delta\nu/\nu = 2.57 \pm 0.26 \times 10^{-15}$, consistent with the theoretical prediction.