Lecture Nine: Theory of Relativity

9.1 Parallel Transport

How do the components of a vector change as we move along a curve? Imagine a curve specified by a parameter τ that varies along the curve $\mathcal{C}$, as in the digram below:

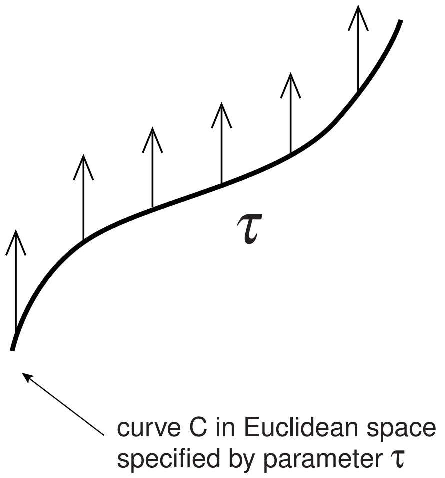

Figure 1:

Now imagine a vector field $A^\mu(\tau)$ defined along the curve. Normally we think of vector fields defined as functions of the coordinates, but I am asking you to think about a vector field defined along a curve (an example would be the spin of a single particle $S_\mu(\tau)$ along the particle's trajectory). In Euclidean space, the condition

$$\frac{dA^\mu}{d\tau} = 0 \tag{1}$$

tells us that the components of the vector field do not change as we move along the curve. In other words if the condition (1) is satisfied, then the vector is 'parallel transported' along the curve $\mathcal{C}$ (the components of A^μ are fixed as we move along the curve, as shown in the diagram).

Equation (1) can't express parallel transport in non-Euclidean spaces because the quantity $dA^\mu/d\tau$ is not a tensor and so depends on the choice of coordinates.

The obvious generalization is to use the covariant derivative with respect to the parameter τ that specifies the curve (not the covariant derivative with respect to the coordinates, because the vector field is defined only along the curve $\mathcal{C}$).

$$\frac{DA^\mu}{D\tau} = \frac{dA^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} A^\nu \frac{dx^\lambda}{d\tau} = 0,$$

or for a covariant vector field

$$\frac{DA_\mu}{D\tau} = \frac{dA_\mu}{d\tau} + \Gamma^\lambda_{\mu\nu} A_\lambda \frac{dx^\nu}{d\tau} = 0.$$

If these equations are satisfied, then they will be true in all coordinate systems and so they generalise the concept of parallel transport to non-Euclidean spaces. We can see this explicitly, since the 'parallel transport' condition tells us that

$$\frac{dA^\mu}{d\tau} = -\Gamma^\mu_{\nu\lambda} A^\nu \frac{dx^\lambda}{d\tau}. \tag{2}$$

In other words, if we specify the components A^ν at *some arbitrary point* τ along the curve, equation (2) fixes the components of A^μ *along the entire length of the curve.* Are you worried about whether the transportation is really parallel? Think about an infinitessimal displacement of the vector field. For a small displacement we can choose *locally* Euclidean coordinates in which the Γ's vanish and so setting the covariant derivative equal to zero describes an infinitessimal displacement which keeps the vector parallel ($dA^\mu = 0$).

Now what happens if we parallel transport a vector field around a closed curve $\mathcal{C}$? Define an arbitrary surface A bounding the curve $\mathcal{C}$ and break the area A up into lots of little closed curves $\mathcal{C}_N$, as in the diagram below.

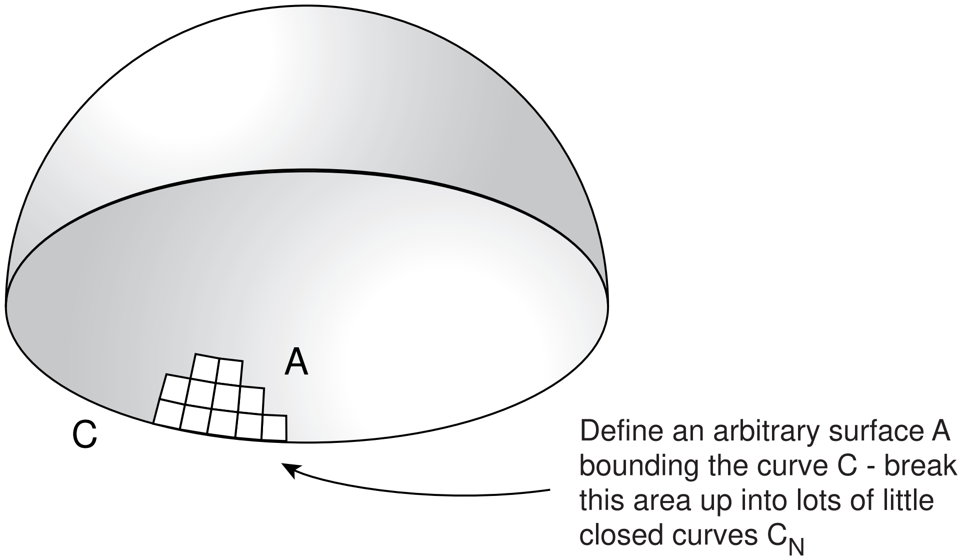

Figure 2:

The change in the components of A_μ around the closed curve $\mathcal{C}$ is

$$\Delta A_\mu = \sum_N (\Delta A_\mu)_N$$

where $(\Delta A_\mu)_N$ is the change in A_μ around a little closed curve. This follows because the changes in ΔA_μ around any of the interior closed curves cancel, leaving just the contributions around the outer edges that bound the curve $\mathcal{C}$.

Now let's calculate the $(\Delta A_\mu)_N$. Expand $\Gamma^\lambda_{\mu\nu}(x)$ in a Taylor series about an arbitrary point $X = x(\tau_0)$

$$\Gamma^\lambda_{\mu\nu}(x) = \Gamma^\lambda_{\mu\nu}(X) + (x^\rho - X^\rho)\frac{\partial \Gamma^\lambda_{\mu\nu}(X)}{\partial X^\rho} + \dots$$

and also expand $A_\mu(\tau)$

$$A_\mu(\tau) = A_\mu(\tau_0) + \Gamma^\lambda_{\mu\nu}(X)(x^\nu - X^\nu)A_\lambda(\tau_0) + \dots$$

Now substitute these expansions into the equation for parallel transport

$$\frac{dA_\mu}{d\tau} = \Gamma^\lambda_{\mu\nu} A^\lambda \frac{dx^\nu}{d\tau},$$

and integrate

$$\begin{aligned} A_\mu(\tau) = A_\mu(\tau_0) \quad + \quad & \int_{\tau_0}^{\tau} \left[\Gamma^\lambda_{\mu\nu}(X) + (x^\rho - X^\rho) \frac{\partial \Gamma^\lambda_{\mu\nu}(X)}{\partial X^\rho} \right] \\ & \left[A_\lambda(\tau_0) + A_\sigma(\tau_0) \Gamma^\sigma_{\lambda\rho}(X)(x^\rho - X^\rho) \right] \frac{dx^\nu}{d\tau} d\tau. \end{aligned}$$

Expanding the terms in brackets and retaining terms to first order in $x^\rho - X^\rho$,

$$\begin{aligned} A_\mu(\tau) &= A_\mu(\tau_0) + \Gamma^\lambda_{\mu\nu}(X) A_\lambda(\tau_0) \int_{\tau_0}^{\tau} \frac{dx^\nu}{d\tau} d\tau \\ &+ \left[\frac{\partial \Gamma^\sigma_{\mu\nu}(X)}{\partial X^\rho} + \Gamma^\sigma_{\lambda\rho}(X) \Gamma^\lambda_{\mu\nu}(X) \right] A_\sigma(\tau_0) \int_{\tau_0}^{\tau} (x^\rho - X^\rho) \frac{dx^\nu}{d\tau} d\tau. \end{aligned} \tag{3}$$

If we integrate the coordinate differentials around a closed loop,

$$\oint dx^\nu = 0$$

and so the first integral in (3) vanishes giving,

$$\Delta A_\mu = \left[\frac{\partial \Gamma^\sigma_{\mu\nu}(X)}{\partial X^\rho} + \Gamma^\sigma_{\lambda\rho} \Gamma^\lambda_{\mu\nu} \right] A_\sigma(\tau_0) \oint x^\rho dx^\nu,$$

and interchanging the indices ν and ρ

$$\Delta A_\mu = \left[\frac{\partial \Gamma^\sigma_{\mu\rho}(X)}{\partial X^\nu} + \Gamma^\sigma_{\lambda\nu} \Gamma^\lambda_{\mu\rho} \right] A_\sigma(\tau_0) \oint x^\nu dx^\rho.$$

Using the relation

$$\oint x^\rho dx^\nu = - \oint x^\nu dx^\rho$$

we see that

$$\Delta A_\mu = \frac{1}{2} R^\sigma_{\mu\nu\rho} A_\sigma \oint x^\rho dx^\nu, \tag{4}$$

where $R^\sigma_{\mu\nu\rho}$ is the *curvature tensor*,

$$R^\lambda_{\mu\nu\kappa} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta}.$$

Equation (4) tells us that the components of A_μ will stay constant after parallel transportation around a small closed curve centred on X *if and only if* the curvature tensor vanishes at the point X. So, returning to our construction of $(\Delta A_\mu)_N$, the vector A_μ will not change on parallel transportation around the entire closed curve $\mathcal{C}$ if the curvature tensor $R^\sigma_{\mu\nu\rho}$ vanishes over the entire area $\mathcal{A}$ bounding the curve.

As an example, consider the parallel transportation of a vector around the closed triangle ABC on the surface of a sphere:

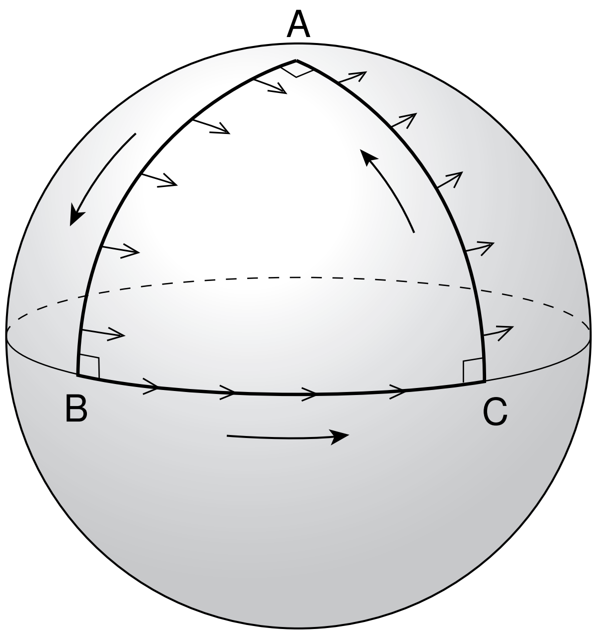

Figure 3:

It is self evident that the vector components change direction after parallel transportation around the triangle.

9.2 Curvature of a spherical surface: a worked example

The metric of the surface of a sphere in spherical polar coordinates is

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2. \tag{5}$$

To get you used to handling problems involving curved spaces I would like you to calculate the components of the affine connection starting from the metric (5). The definition of the affine connection is

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right\},$$

and in two dimensions there are six independent connection components:

$$\Gamma^1_{11}, \quad \Gamma^1_{12}, \quad \Gamma^1_{22}, \quad \Gamma^2_{11}, \quad \Gamma^2_{12}, \quad \Gamma^2_{22}.$$

I have worked out these components in full so that you can check your calculations:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{11} \left\{ \frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right\} = 0, \\
\Gamma^1_{12} &= \frac{1}{2} g^{11} \left\{ \frac{\partial g_{11}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^1} \right\} = 0, \\
\Gamma^1_{22} &= \frac{1}{2} g^{11} \left\{ \frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1} \right\} = -\frac{1}{2} g^{11} \frac{\partial g_{22}}{\partial x^1},
\end{aligned}$$

$$
\begin{aligned}
\Gamma^2_{11} &= \frac{1}{2}g^{22}\left\{\frac{\partial g_{12}}{\partial x^2}+\frac{\partial g_{12}}{\partial x^2}-\frac{\partial g_{11}}{\partial x^2}\right\}=0,\\
\Gamma^2_{12} &= \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^1}+\frac{\partial g_{12}}{\partial x^2}-\frac{\partial g_{11}}{\partial x^2}\right\}=\frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial x^1},\\
\Gamma^2_{22} &= \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^2}+\frac{\partial g_{22}}{\partial x^2}-\frac{\partial g_{22}}{\partial x^2}\right\}=0.
\end{aligned}
$$

So, you should end up with only two non-zero components

$$
\begin{aligned}
\Gamma^1_{22} &= -\frac{1}{2a^2}a^2 2\sin\theta\cos\theta=-\sin\theta\cos\theta,\\
\Gamma^2_{12} &= \frac{1}{2a^2\sin^2\theta}2a^2\sin\theta\cos\theta=\frac{\cos\theta}{\sin\theta}.
\end{aligned}
$$

The curvature tensor is

$$
R_{\lambda\mu\nu\kappa}=\frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa\partial x^\mu}-\frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa\partial x^\lambda}-\frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu\partial x^\mu}+\frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu\partial x^\lambda}\right]+g_{\eta\sigma}\left[\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa}-\Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right],
$$

and in two dimensions the symmetry properties of this tensor means that there is only *one* independent component. We can take this to be R_{1212}, so fortunately we only have to calculate this single component:

$$
\begin{aligned}
R_{1212} &= \frac{1}{2}\left[\frac{\partial^2 g_{11}}{\partial (x^2)^2}-\frac{\partial^2 g_{21}}{\partial x^2\partial x^1}-\frac{\partial^2 g_{12}}{\partial x^1\partial x^2}+\frac{\partial^2 g_{22}}{\partial (x^1)^2}\right]+g_{\eta\sigma}\left[\Gamma^\eta_{11}\Gamma^\sigma_{22}-\Gamma^\eta_{21}\Gamma^\sigma_{21}\right]\\
&= \frac{1}{2}\left[\frac{\partial^2 g_{22}}{\partial (x^1)^2}\right]+g_{11}\left[\Gamma^1_{11}\Gamma^1_{22}-\Gamma^1_{12}\Gamma^1_{12}\right]+g_{22}\left[\Gamma^2_{11}\Gamma^2_{22}-\Gamma^2_{21}\Gamma^2_{21}\right]\\
&= -a^2\sin^2\theta.
\end{aligned}
$$

So, you have now understood the origin of Gauss's curvature parameter that I introduced in Section 5. Gauss's curvature parameter is just

$$
K=-\frac{R_{1212}}{g}=\frac{a^2\sin^2\theta}{a^4\sin^2\theta}=\frac{1}{a^2},
$$

and if you write out R_{1212} in full in terms of derivatives of the $g_{\mu\nu}$ you will recover the formidably complicated equation for the Gaussian curvature K of Section 5.

9.3 Surface of a cylinder

Instead of a spherical surface, consider the surface of a cylinder of radius a. The metric of the surface in cylindrical polar coordinates is

$$
ds^2=a^2d\theta^2+dz^2,
$$

and it is obvious that this two-dimensional space is spatially flat because we can transform the metric into the form

$$
ds^2=dx^2+dz^2,
$$

by the coordinate transformation $x = a\theta$. It therefore follows that the curvature of the cylindrical surface vanishes. Furthermore, since the curvature tensor vanishes everywhere, the components of a vector will remain unchanged if the vector is parallel transport around any closed curve on the cylinder. For example, this is evident on parallel transportation of a vector around the closed rectangle ABCD in the figure below:

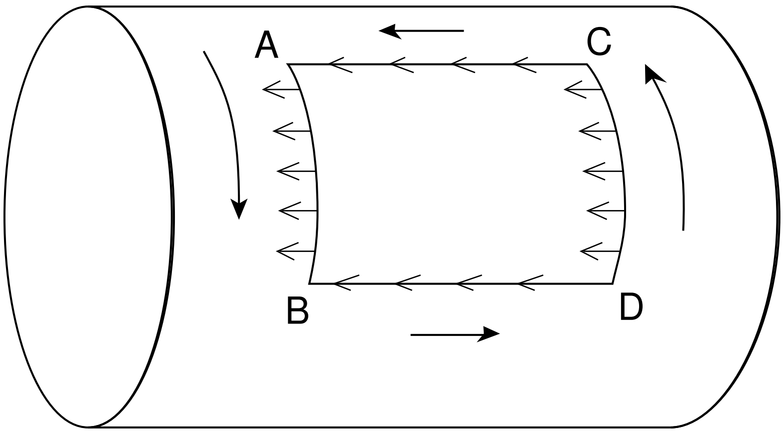

Figure 4: