Lecture Ten: Theory of Relativity

10.1 Geodesic Deviation

Consider the motion of two freely falling particles on nearby trajectories,

$$x^\mu(\tau),$$
$$x^\mu(\tau) + \delta x^\mu(\tau).$$

Each of these trajectories must obey the geodesic equations of motion:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0, \tag{1}$$

$$\frac{d^2\left[x^\mu + \delta x^\mu\right]}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\left(x^\kappa + \delta x^\kappa\right)\frac{d}{d\tau}\left(x^\nu + \delta x^\nu\right)\frac{d}{d\tau}\left(x^\lambda + \delta x^\lambda\right) = 0. \tag{2}$$

Expand (2) in a Taylor series to first order in δx^λ,

$$\frac{d^2x^\mu}{d\tau^2} + \frac{d^2\delta x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial x^\rho}\delta x^\rho\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{d\delta x^\lambda}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\frac{d\delta x^\nu}{d\tau} = 0,$$

and subtract (1)

$$\frac{d^2\delta x^\mu}{d\tau^2} + \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial x^\rho}\delta x^\rho\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + 2\Gamma^\mu_{\mu\lambda}\frac{dx^\nu}{d\tau}\frac{d\delta x^\lambda}{d\tau} = 0.$$

This is quite a complicated equation to work with, but we can simplify it considerably by defining a set of *local* Euclidean coordinates ξ^μ. In such a coordinate system, we know that the Γ's vanish. The derivatives of Γ with respect to the coordinates will not necessarily vanish (as we will see below). In the locally Euclidean coordinate system (2) simplifies to

$$\frac{d^2\delta\xi^\mu}{d\tau^2} + \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial\xi^\rho}\delta\xi^\rho\frac{d\xi^\nu}{d\tau}\frac{d\xi^\lambda}{d\tau} = 0. \tag{3}$$

The covariant derivative of $\delta\xi^\mu$ with respect to τ is

$$\frac{D\delta\xi^\mu}{D\tau} = \frac{d\delta\xi^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{d\xi^\lambda}{d\tau}\delta\xi^\nu,$$

now don't set the Γ's equal to zero at this stage because we want to differentiate again and this will introduce a derivative of the Γ's.

$$\frac{D^2\delta\xi^\mu}{D\tau^2} = \frac{d^2\delta\xi^\mu}{d\tau^2} + \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial\xi^\sigma}\frac{d\xi^\sigma}{d\tau}\frac{d\xi^\lambda}{d\tau}\delta\xi^\nu.$$

I have now set the Γ's equal to zero in this equation but retained the first derivative of Γ. Now substitute into equation (3),

$$\frac{D^2\delta\xi^\mu}{D\tau^2} + \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial\xi^\rho}\delta\xi^\rho\frac{d\xi^\nu}{d\tau}\frac{d\xi^\lambda}{d\tau} - \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial\xi^\sigma}\frac{d\xi^\sigma}{d\tau}\frac{d\xi^\lambda}{d\tau}\delta\xi^\nu = 0,$$

i.e.,

$$\frac{D^2\delta\xi^\mu}{D\tau^2} + \left(\frac{\partial\Gamma^\mu_{\sigma\lambda}}{\partial\xi^\rho} - \frac{\partial\Gamma^\mu_{\rho\lambda}}{\partial\xi^\sigma}\right)\frac{d\xi^\sigma}{d\tau}\frac{d\xi^\lambda}{d\tau}\delta\xi^\rho = 0.$$

But, in the locally Euclidean coordinate system in which the Γ's are zero, the curvature tensor is simply

$$R^\mu_{\lambda\sigma\rho} = \frac{\partial\Gamma^\mu_{\sigma\lambda}}{\partial\xi^\rho} - \frac{\partial\Gamma^\mu_{\lambda\rho}}{\partial\xi^\sigma}.$$

So, we can write a generally covariant equation describing *geodesic deviation*,

$$\frac{D^2\delta x^\mu}{D\tau^2} + R^\mu_{\lambda\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\rho = 0. \tag{4}$$

Notice that I have replaced the locally Euclidean coordinates by arbitrary coordinates x^μ. I can do this because equation (4) is a tensorial equation and so is true in any coordinate system. The locally Euclidean coordinate system that I introduced is just a device to simplify the algebra in deriving the tensorial equation (4).

The physical meaning of equation (4) is straightforward. If space is curved ($R^\mu_{\lambda\sigma\rho} \neq 0$) two neighbouring geodesics (or freely falling particles) will either converge or diverge, depending on the local curvature. For example, the two neighbouring geodesics AB and AC on the surface of the sphere converge as we approach the point A at the pole because the surface is positively curved.

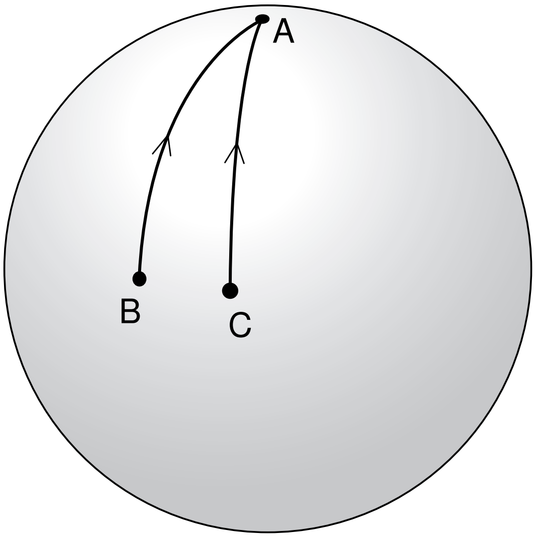

Figure 1:

Equation (4) allows us to compute the rates of convergence or divergence of neighbouring geodesics for Riemannian spaces of arbitrary complexity. All you have to do is to compute the curvature tensor at each point from the metric.

10.2 The energy-momentum tensor

Imagine that we have a set of non-interacting particles (this is commonly called "dust" in the literature). We can characterize such a distribution by

- $\rho_0(\mathbf{x}) \Rightarrow$ the proper density measured by an observer comoving with the local flow of the dust.
- $u^\mu(\mathbf{x}) = dx^\mu/d\tau \Rightarrow$ the four-velocity of the flow at point x.

Now, let's look at the quantity $\rho_0(\mathbf{x})$. This quantity is *not* a scalar. In a Lorentz frame S at rest with respect to the flow we can write

$$\rho_0(\mathbf{x}) = m_0 n_0,$$

where m_0 is the rest mass of the particles (for simplicity I have assumed that all the particles have the same mass) and n_0 is the density of the particles in this frame. Now transform to a Lorentz frame S' so that S is now moving at speed v relative to S'. You can see what will happen from the following figure

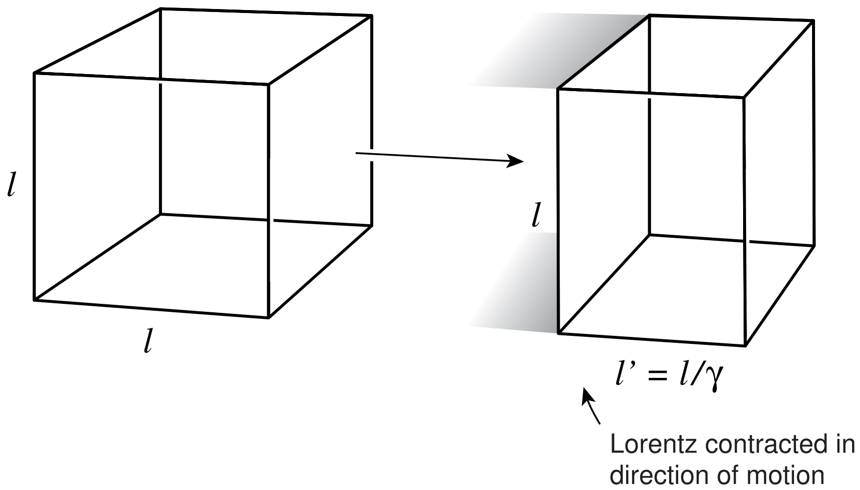

Figure 2:

In frame S', the box appears Lorentz contracted in the direction of motion. The number of particles in the box must be the same (all we have done is change the inertial frame and so if the box contains N non-interacting particles in frame S it will contain N particles in frame S'. However, the mass of each particle is boosted by a factor of γ and the volume of the box is smaller by a factor of γ because of Lorentz contraction. The mass density in the frame S' is therefore

$$\begin{aligned}\rho' &= m'n' \\ &= \gamma m_0 \frac{N}{\ell^2 \ell'} \\ &= \gamma^2 m_0 \frac{N}{\ell^3} \\ &= \gamma^2 \rho_0\end{aligned}$$

Evidently $\rho_0(x)$ does not transforms as scalar. But it *does* transform like the 00 component of a *second rank tensor.*

The quantities ρ_0 and u^μ characterizing the dust define a second rank tensor

$$T^{\mu\nu} = \rho_0(x) u^\mu u^\nu. \tag{1}$$

Recall that we can write the four velocity in any frame as $\gamma(c, \mathbf{v})$ where $v^i = dx^i/dt$. So it is straightforward to write down the components of this tensor from the definition (1)

$$\begin{aligned} T^{00} &= \rho_0 u^0 u^0 = \gamma^2 \rho_0 c^2, \\ T^{0i} &= T^{i0} = \rho_0 u^0 u^i = \gamma^2 \rho_0 c v^i, \\ T^{ij} &= T_{ji} = \rho_0 u^i u^j = \gamma^2 \rho_0 v^i v^j. \end{aligned}$$

You can see from the first of these equations that $\rho(x)$ can be identified with the T^{00} component of a second rank tensor. But what does this tensor actually mean? From the definition (1), we can interpret the components as follows:

- $T^{\circ\circ}$ is the energy density of the particles,
- T^{0i} is the momentum density in the i-direction
- T^{ii} is the rate of flow of the i-component of the momentum per unit area in the i-direction (*i.e.* the *pressure* in the i-direction),
- T^{ij} is the rate of flow of the i-component of the momentum per unit area in the j-direction (*i.e.* the *viscous stress*).

Because of these identifications, the tensor $T^{\mu\nu}$ is called the *energy-momentum* tensor, or sometimes the *stress-energy* tensor.

10.3 Conservation of energy and momentum

Let's first look at how to express energy conservation in Special Relativity. Construct the quantity

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu}.$$

Set $\mu = 0$,

$$\begin{aligned} \frac{\partial T^{0\nu}}{\partial x^\nu} &= \frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i}, \\ &= \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v^i)}{\partial x^i}. \end{aligned}$$

But you will recognise this as the (relativistic) equation of continuity. Conservation of mass-energy therefore requires that

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0.$$

Now look at the component $\partial T^{i\nu}/\partial x^\nu$,

$$\frac{\partial T^{i\nu}}{\partial x^\nu} = \frac{\partial T^{i0}}{\partial t} + \frac{\partial T^{ij}}{\partial x^j},$$

and evaluate this for $i = 1$,

$$\frac{\partial(\rho v^1)}{\partial t} + \frac{\partial}{\partial x^1}(\rho v^1 v^1) + \frac{\partial}{\partial x^2}(\rho v^1 v^2) + \frac{\partial}{\partial x^3}(\rho v^1 v^3)$$
$$= \frac{\partial \rho}{\partial t} v^1 + \rho \frac{\partial v^1}{\partial t} + \rho v^1 \frac{\partial v^1}{\partial x^1} + \rho v^2 \frac{\partial v^1}{\partial x^2} + \rho v^3 \frac{\partial v^1}{\partial x^3} + v^1 \frac{\partial(\rho v^1)}{\partial x^1} + v^1 \frac{\partial(\rho v^2)}{\partial x^2} + v^1 \frac{\partial(\rho v^3)}{\partial x^3},$$

and substitute for $\partial\rho/\partial t$ from the equation of continuity,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v^1)}{\partial x^1} - \frac{\partial(\rho v^2)}{\partial x^2} - \frac{\partial(\rho v^3)}{\partial x^3},$$

we get

$$\frac{\partial T^{i\nu}}{\partial x^\nu} = \rho \frac{\partial v^i}{\partial t} + \rho\, (\mathbf{v} \cdot \boldsymbol{\nabla})\, v^i.$$

But in the absence of forces

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \right) = 0$$

represents conservation of momentum. In fluid mechanics this equation is called the Navier-Stokes equation.

Thus, in special relativity, conservation of energy and momentum can be summarised as

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0, \tag{2}$$

and this equation gives us the equations of fluid dynamics (the continuity and Navier-Stokes equation in the absence of external forces).

Equation (2) is not a tensorial equation – we need to generalize it to describe energy-momentum conservation in General Relativity. To generalise this equation, simply replace the derivative with respect to the coordinates with a covariant derivative:

$$\boxed{T^{\mu\nu}{}_{;\nu} = 0.}$$

This equation is covariant and expresses conservation of energy and momentum in General Relativity.

10.4 Energy Momentum Tensor of a perfect fluid

A *perfect* fluid is described by two scalar functions

$$\begin{aligned} \rho &\Rightarrow \text{the density} \\ p &\Rightarrow \text{the isotropic pressure} \end{aligned}$$

In a frame in which the fluid is at rest, symmetry requires that the energy-momentum tensor has components

$$\begin{aligned} T^{00} &= \rho, \\ T^{i0} &= T^{0i} = 0, \\ T^{ij} &= p\delta_{ij}. \end{aligned}$$

In other words a perfect fluid has no anisotropic stresses and is described by a diagonal energy-momentum tensor.

Now transform to the 'laboratory' frame in which the fluid is moving at speed $\mathbf{v}$. The components of $T^{\mu\nu}$ transform as a second rank tensor and so transform as

$$T^{\mu\nu} = \Lambda^{\mu}_{\alpha}(\mathbf{v})\Lambda^{\nu}_{\beta}(\mathbf{v})T^{\alpha\beta},$$

where the Λ's are the components of the Lorentz transformation matrix. The components of $T^{\mu\nu}$ therefore transform as

$$\begin{aligned} T^{00} &= \gamma^2\left(\rho + \rho v^2\right), \\ T^{i0} &= \gamma^2\left(p + \rho\right)v^i, \\ T^{ij} &= \gamma^2\left(p + \rho\right)v^i v^j + p\delta_{ij}. \end{aligned}$$

or, if we write this in tensorial form,

$$\boxed{T^{\mu\nu} = \left(p + \rho\right)u^{\mu}u^{\nu} - pg^{\mu\nu}.}$$

This is the energy-momentum tensor of a perfect fluid.