Lecture Eleven: Theory of Relativity

11.1 The Gravitational Field Equations

We are now in a position to tackle the problem of deriving a set of field equations for gravity. Let's recall a few results from previous lectures:

- We showed that in a weak gravitational field

$$g_{00} = \left(1 + \frac{2\phi}{c^2}\right),$$

  (Henceforth in these lecture notes I will (usually) set $c = 1$.)

- We have constructed an energy-momentum tensor

$$T^{\mu\nu},\ T_{\mu\nu},$$

  with a T_{00} component which, for 'dust', measures the mass density

$$T_{00} = \rho.$$

- The gravitational field equation of Newtonian gravity is

$$\nabla^2\phi = 4\pi G\rho,$$

  i.e., for a weak gravitational field

$$\nabla^2 g_{00} = 8\pi G T_{00}. \tag{1}$$

- The last equation suggests that the field equations are tensorial equations of rank 2, of the form

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2}$$

  where the tensor $G_{\mu\nu}$ must be consistent with the Newtonian limit (1).

But what is the form of $G_{\mu\nu}$? We have a few clues:

- $G_{\mu\nu}$ must involve second derivatives of the metric tensor so that in the weak field limit $G_{00} \to \nabla^2 g_{00}$.
- The vanishing of a gravitational field is equivalent to the vanishing of the curvature tensor, which as we know involves the second derivatives of $g_{\mu\nu}$. This suggests that $G_{\mu\nu}$ is related to rank two tensors constructed from the curvature tensor.
- $T_{\mu\nu}$ is symmetric and so $G_{\mu\nu}$ must be symmetric.
- $T_{\mu\nu}$ is conserved, $T_{\mu\nu;\nu} = 0$, hence $G_{\mu\nu}$ must also be conserved $G_{\mu\nu;\nu} = 0$.

As a guess, let's write down a rank 2 tensor constructed from the curvature tensor:

$$\begin{array}{ccccccccc}
 & & & \text{Ricci} & & & & \text{Curvature} \\
 & & & \text{tensor} & & & & \text{Scalar} \\
 & & & \downarrow & & & & \downarrow \\
G_{\mu\nu} & = & \alpha & R_{\mu\nu} & + & \beta & g_{\mu\nu} & R \\
 & & \nwarrow & & \nearrow & & & \\
 & & & \text{constants} & & & &
\end{array}$$

This guess automatically satisfies the first three conditions. The tensor $G_{\mu\nu}$ involves just the metric tensor and its first two derivatives. The tensor $G_{\mu\nu}$ is symmetric and it vanishes if all components of the curvature tensor vanish.

Now let's make use of the condition that $G_{\mu\nu}$ satisfies a conservation equation. Raise an index

$$\begin{aligned}
G^{\kappa}_{\nu} &= g^{\kappa\mu}G_{\mu\nu} = \alpha g^{\kappa\mu}R_{\mu\nu} + \beta g^{\kappa\mu}g_{\mu\nu}R \\
&= \alpha R^{\kappa}_{\nu} + \beta\delta^{\kappa}_{\nu}R,
\end{aligned}$$

and take the covariant derivative,

$$G^{\kappa}_{\nu;\kappa} = \alpha R^{\kappa}_{\nu;\kappa} + \beta\delta^{\kappa}_{\nu}R_{;\kappa}.$$

Now recall the Bianchi identity

$$\left(R^{\mu}_{\eta} - \frac{1}{2}\delta^{\mu}_{\eta}R\right)_{;\mu} = 0,$$

this gives

$$G^{\kappa}_{\nu;\kappa} = \left(\frac{1}{2}\alpha + \beta\right)R_{;\nu},$$

and so if $G_{\mu\nu}$ satisfies a conservation equation

$$G^{\kappa}_{\nu;\kappa} = 0, \quad \text{either} \quad \left\{\begin{array}{l} \beta = -\frac{1}{2}\alpha, \\ \\ R_{;\nu} = 0. \end{array}\right.$$

We can easily rule out the idea that $R_{;\nu}$ is always equal to zero starting from the field equations (2).

$$\begin{aligned}
G^{\kappa}_{\kappa} &= 8\pi G T^{\kappa}_{\kappa} \\
&= \alpha R^{\kappa}_{\kappa} + \beta\delta^{\kappa}_{\kappa}R \\
&= \alpha R + \beta R \\
&= \alpha' R, \qquad\qquad \text{where } \alpha' = \alpha + \beta.
\end{aligned}$$

So, taking the covariant derivative of this last relation

$$\alpha' R_{;\nu} = 8\pi G T^{\kappa}_{\kappa;\nu} = 0, \tag{3}$$

But (3) requires that

$$\frac{\partial T^{\kappa}_{\kappa}}{\partial x^{\nu}} = 0 \quad \text{always, which cannot be satisfied in general since it requires } \partial\rho_0/\partial x = 0.$$

We must therefore select the solution

$$\beta = -\frac{1}{2}\alpha$$

and so the field equations must look like

$$\alpha\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu}. \qquad (4)$$

To fix the constant α, we can look at the weak field limit of equations (4).

For non-relativistic matter,

$$T_{ij} \ll T_{00} \quad \left(\begin{array}{ccc} T_{00} & \sim & \rho_0 c^2 \\ T_{ij} & \sim & \rho_0 v^2 \end{array}\right)$$

and for weak gravitational fields

$$g_{\mu\nu} \approx \eta_{\mu\nu},$$

i.e. the metric tensor is nearly that of Minkowski space-time. The field equations look like

$$\alpha\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu},$$

and so if $T_{ij} \approx 0$, the spatial components on the left hand side must be approximately zero, *i.e.*

$$\begin{aligned} R_{ij} &\approx \frac{1}{2}g_{ij}R, \\ &\approx \frac{1}{2}\eta_{ij}R. \end{aligned}$$

The definition of the curvature scalar is

$$\begin{aligned} R &= g^{\mu\kappa}R_{\mu\kappa} \\ &\approx R_{00} - R_{ii} \\ &= R_{00} + \frac{3}{2}R, \qquad \text{using } R_{ij} \approx \frac{1}{2}\eta_{ij}R. \end{aligned}$$

Hence

$$R \approx -2R_{00},$$

and since

$$\begin{aligned} G_{\mu\nu} &= \alpha\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right), \\ G_{00} &= \alpha\left(R_{00} - \frac{1}{2}g_{00}R\right), \\ &\approx \alpha\left(R_{00} + \frac{1}{2}2R_{00}\right) \\ &= 2\alpha R_{00}. \end{aligned}$$

The curvature tensor is, by definition,

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\kappa}\right] + g_{\eta\sigma}\left[\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right].$$

In a weak static field, this expression simplifies significantly because

- all time derivatives vanish,
- products of Γ's can be ignored (they are second order in $g_{\mu\nu}$).
- we know that in the weak field limit,

$$g_{00} \approx (1 + 2\phi).$$

so, since

$$R_{\mu\nu} = g^{\lambda\kappa} R_{\lambda\mu\kappa\nu},$$

the 00 component is

$$R_{00} = g^{\lambda\kappa} R_{\lambda 0 \kappa 0} \approx R_{0000} - R_{i0i0}.$$

But

$$R_{0000} \approx 0, \quad R_{i0j0} = \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^i \partial x^j},$$

and so for a static weak gravitational field,

$$G_{00} = 2\alpha R_{00} = -\alpha \nabla^2 g_{00},$$

i.e.

$$-\alpha \nabla^2 g_{00} = 8\pi G T_{00}.$$

But, in a weak gravitational field the field equations must take the form

$$\nabla^2 g_{00} = 8\pi G T_{00},$$

hence

$$\alpha = -1.$$

Having fixed the constant α, this means that the field equations must have the form

$$\boxed{\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) = -8\pi G T_{\mu\nu}.}$$

These are Einstein's field equations of General Relativity.

11.2 Sign Conventions

There is no accepted system of sign conventions in General Relativity. Different books use different sign conventions for the metric tensor, curvature tensors and for the field equations. We can summarize these sign conventions in terms of three sign factors $S1$, $S2$ and $S3$. These are defined as follows:

$$\begin{aligned}
\eta^{\mu\nu} &= [S1]\,(-1\ \ +1,\ \ +1,\ \ +1) \\
R^{\mu}_{\alpha\beta\gamma} &= [S2]\left(\frac{\partial\Gamma^{\mu}_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial\Gamma^{\mu}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\gamma\alpha} - -\Gamma^{\mu}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\alpha}\right) \\
G_{\mu\nu} &= [S3]\,8\pi G T_{\mu\nu} \\
R_{\mu\nu} &= [S2]\,[S3]\,R^{\alpha}_{\mu\alpha\nu}
\end{aligned}$$

I have chosen to use a convention in which all three sign factors are *negative*. This because of the way that I learned General Relativity, but it is exactly the opposite of the convention used by, for example, Misner, Thorne and Wheeler. My convention is similar to d'Inverno and Rindler, except that I define the curvature tensor with the opposite sign. Here is a summary of the sign conventions used in various books:

	Us	d'Inverno/Rindler	Misner, Thorne, Wheeler	Weinberg
$[S1]$	-	-	+	+
$[S2]$	-	+	+	-
$[S3]$	-	-	+	-