Lecture Twelve: Theory of Relativity

12.1 Are the field equations unique?

In the last lecture we derived the field equations

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = -8\pi G T_{\mu\nu}. \tag{1}$$

I now want to ask whether these field equations are unique. In fact, shortly after Einstein derived these field field equations, he proposed a modification. We have seen that energy-momentum conservation in GR is expressed by

$$T_{\mu\nu;\nu} = 0,$$

and we constructed the right hand side of equation (1) from the curvature tensor to satisfy this relation. But you will recall that the covariant derivative of the metric tensor is equal to zero

$$g_{\mu\nu;\nu} = 0,$$

and so it follows that a *consistent* set of field equations is

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu}\right) = -8\pi G T_{\mu\nu} \qquad (2),$$

where Λ is some new universal constant of Nature. The term Λ is called the *cosmological constant*. Einstein introduced this term because he was unable to construct static models of the Universe from his standard field equations (1). What he found (and we will discuss this in detail in later lectures) was that the standard field equations predicted a Universe that was either expanding or contracting. Einstein did this work in about 1916, when people thought that our Milky Way Galaxy represented the whole Universe, which Einstein represented as a uniform distribution of 'fixed stars'. By introducing Λ, Einstein constructed static models of the Universe (which as we will see are actually unstable). Later, in 1929, Hubble discovered the expansion of the Universe by measuring distances and redshifts to nearby galaxies. The Universe was proved to be expanding and the need for a cosmological constant disappeared. Einstein is reputed to have said that the introduction of the cosmological constant was his 'biggest blunder'.

Nowadays we have a rather different view of the cosmological constant. Recall that the energy momentum tensor of a perfect fluid is

$$T^{\mu\nu} = (p + \rho) u^\mu u^\nu - p g^{\mu\nu}$$

Imagine some type of 'substance' with a strange equation of state $p = -\rho$. This is unlike any kind of substance that you have ever encountered because it has a negative pressure! The energy momentum tensor for this substance is

$$T_{\mu\nu} = -p g_{\mu\nu} = \rho g_{\mu\nu}.$$

Now there are two points to note about this equation. Firstly, the energy momentum tensor of this strange stuff depends only on the *metric* tensor – it is therefore a property of the vacuum itself and we can call ρ the energy density of the *vacuum*. Secondly, the form of $T_{\mu\nu}$ looks just like the cosmological constant term in (2). We can therefore view the cosmological constant as a universal constant that fixes the energy density of the vacuum,

$$\rho_{\text{vac}} = \frac{\Lambda}{8\pi G}.$$

How can we calculate the energy density of the vacuum? This is one of the major unsolved problems in physics. The simplest calculation, involves summing the quantum mechanical zero point energies of all of the fields known in Nature. This gives an answer about 120 orders of magnitude higher than the upper limits on Λ set by cosmology. This is probably the worst theoretical prediction in the history of physics! Nobody knows how to make sense of this result. There must exist some physical mechanism that makes the cosmological constant very small. Some physicists have thought that there must exist a mechanism that makes Λ *exactly* equal to zero. But in the last few years there has been increasing evidence that the cosmological constant is small but non-zero. The strongest evidence comes from observations of distant Type Ia supernovae that indicate that the Universe is actually accelerating rather than decelarating. Normally, one would have thought that the gravity of matter in the Universe would cause the expansion to slow down (perhaps even eventually halting the expansion and causing the Universe to collapse). But if the cosmological constant is non-zero, the negative pressure of the vacuum can cause the Universe to accelerate.

Whether the supernova observations are right or not is an area of active research. The theoretical problem of explaining the value of the cosmological constant is, as I have said, one of the great challenges of theoretical physics. My bet is that we will need a fully developed theory of quantum gravity (perhaps superstring theory) before we can understand Λ.

12.2 Brans-Dicke Theory

In the previous section, I described a relatively simple (but theoretically profound) modification of the Einstein field equations. This shows you that Einstein's field equations are not unique. It is also worth showing you that it is possible to make more radically different theories of gravity. In this section, I will discuss one of the most important 'alternative' theories – the Brans-Dicke theory. I will present just a sketch of the theory, not a detailed derivation, so that you can get an idea of how the theory works. I hope that this will also give you a better understanding of General Relativity itself.

In deriving the Einstein field equations, we started with the principle of equivalence and found a rank two tensor theory of gravity that agreed with Newton's theory in the limit of weak gravitational fields. Brans and Dicke set about finding a fundamentally different type of theory. They took the principle of equivalence as a starting point, but sought a consistent *scalar-tensor* theory of gravity. Instead of treating the gravitational constant G as a constant of nature, Brans and Dicke introduced a scalar field ϕ which determines the strength of G, *i.e.* the scalar field ϕ determines the coupling strength of matter to gravity. The key ideas of the theory are:

- matter (represented by the energy-momentum tensor $T^{\mu\nu}_M$) and a coupling constant λ fix the scalar field ϕ,

- the scalar field ϕ fixes the value of G,
- the gravitational field equations relate the curvature to the energy-momentum tensors of the scalar field and matter.

The equation for the scalar field in this theory is

$$\Box^2 \phi = 4\pi \lambda T^{\mu}_{M\mu},$$

and the gravitational field equations are

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = -\frac{8\pi}{\phi} \left[T_{M\mu\nu} + T_{\Phi\mu\nu} \right],$$

where $T_{M\mu\nu}$ is the energy-momentum tensor of the matter and $T_{\Phi\mu\nu}$ is the energy-momentum tensor of the scalar field ϕ. Invoking the principle of equivalence fixes $T_{\Phi\mu\nu}$ and leads to a consistent set of field equations:

$$\Box^2 \phi = \frac{8\pi}{3 + 2\omega} T^{\mu}_{M\mu}$$

$$\left\{ \begin{array}{ll} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R & = \frac{8\pi}{\phi} T_{M\mu\nu} \\ & -\frac{\omega}{\phi^2} \left(\phi_{;\mu} \phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \phi_{;\rho} \phi_{;\rho} \right) \\ & -1/\phi \left(\phi_{;\mu;\nu} - g_{\mu\nu} \Box^2 \phi \right) \end{array} \right\} \longrightarrow \text{ where } \phi \approx \frac{1}{G} + \mathcal{O}\left(\frac{1}{\omega}\right).$$

In these equations, I have written the coupling constant λ in terms of the parameter ω, so that the scalar field ϕ is nearly equal to the inverse of G for large values of ω. This is convenient because the field equations can then be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{M\mu\nu} + \mathcal{O}\left(\frac{1}{\omega}\right),$$

and so you can see from these equations that Brans-Dicke theory reduces to Einstein's theory in the limit that $\omega \to \infty$.

The Brans-Dicke theory is interesting because it shows that it is possible to construct alternative theories that are consistent with the principle of equivalence. Einstein's theory is beautiful and simple, but it is not unique. One must therefore look to experiment to find out which theory is correct. One of the features of the Brans-Dicke model is that the effective gravitational 'constant' G varies with time because it is fixed by the scalar field ϕ. A variation of G would affect the orbits of the planets altering, for example, the dates of solar eclipses (which can be checked against historical records). I won't go into all of the constraints now (we will look at some in later lectures), but a reasonably conservative conclusion from experiments is that $\omega \geq 500$. Einstein's theory does seem to be the correct theory of gravity, at least at low energies.

12.3 More on Einstein's field equations

Einstein's field equations are

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = -8\pi G T_{\mu\nu}, \tag{1}$$

and in 4-dimensions we have 10 independent components of the $g_{\mu\nu}$ and so from (1) in Einstein's theory we have 10 field equations. (Compare this to Newtonian gravity where we have only one gravitational field equation). Furthermore, the Einstein field equations are *non-linear* in the $g_{\mu\nu}$ (whereas Newtonian gravity is linear in the field ϕ). Einstein's theory thus involves lots of non-linear differential equations, and so it should come as no surprise that the theory is complicated.

There is another interesting aspect of the theory. Contract the field equations (1),

$$\begin{aligned}\left(g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R\right) &= -8\pi G T_{\mu\nu}g^{\mu\nu} \\ \left(R - \frac{1}{2}4R\right) &= -R = -8\pi G T^{\mu}_{\mu}.\end{aligned}$$

So, an entirely equivalent form of the field equations is

$$R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\mu}_{\mu}\right). \tag{2}$$

It follows from (2) that the field equations in *empty space* are

$$\boxed{R_{\mu\nu} = 0.}$$

From this simple equation, we can immediately establish a profound result. Look at the number of field equations as a function of the number of space-time dimensions

No. of Dimensions	2	3	4
No. of field equations	3	6	10
No. of independent components of $R_{\lambda\mu\nu\kappa}$	1	6	20
	$\underbrace{\hphantom{1 \quad\quad 6}}$ Vanishing of $R_{\mu\nu}$ guarantees vanishing of $R_{\lambda\mu\nu\kappa}$		

You see that in two or three dimensions, the field equations in empty space guarantee that the full curvature tensor must vanish. In four dimensions, however, we have 10 field equations but 20 independent components of the curvature tensor. It is therefore possible to satisfy the field equations in empty space with a non-vanishing curvature tensor. But remember that a non-vanishing curvature tensor represents a non-vanishing gravitational field. We conclude that *it is only in 4 dimensions or more that gravitational fields can exist in empty space.*