Lecture Thirteen: Theory of Relativity

13.1 Static, isotropic, metric

In this section, I want to find a static, isotropic, metric. What does this mean? Starting from the metric interval,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

we want to find a set of coordinates x^μ in which the $g_{\mu\nu}$ do not depend on time t (*i.e.* the metric is static) and in which ds^2 depends *only* on rotational invariants of the spatial coordinates (*i.e.* the metric is isotropic). We can form the following rotational invariants from the coordinates:

$$\left.\begin{array}{c} d\mathbf{x}^2 \\ \mathbf{x}\cdot d\mathbf{x} \\ \mathbf{x}^2 \equiv r^2 \end{array}\right\} \text{ ($\mathbf{x}$ are the spatial coordinates)}$$

and so the general form of a static, isotropic, metric must be

$$ds^2 = A(r)dt^2 - B(r)dt\, \mathbf{x}\cdot d\mathbf{x} - C(r)(\mathbf{x}\cdot d\mathbf{x})^2 - D(r)d\mathbf{x}^2, \tag{1}$$

where A, B, C and D are arbitrary function of the radius r. The metric must have the form (1) because we seek a metric that is quadratic in the coordinate differentials that must be rotationally invariant. Now transform to spherical polar coordinates

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi, \\ x^2 &= r\sin\theta\sin\phi, \\ x^3 &= r\cos\theta. \end{aligned}$$

Since

$$\begin{aligned} \mathbf{x}^2 &= r^2, \quad \mathbf{x}\cdot d\mathbf{x} = r dr, \\ d\mathbf{x}^2 &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \end{aligned}$$

the general metric (1) looks like

$$ds^2 = A(r)dt^2 - B(r)dt\, rdr - C(r)r^2dr^2 - D(r)\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right] \tag{2}$$

in spherical polar coordinates. Now define a new time coordinate

$$\begin{aligned} t' &= t + \Phi(r), \\ dt' &= dt + d\Phi(r), \end{aligned}$$

and lets look at the first two terms in (2)

$$A(r)\left[dt' - d\Phi\right]^2 - B(r)\left[dt' - d\Phi\right] rdr$$

$$= A(r)dt'^2 + A(r)\left(\frac{d\Phi}{dr}\right)^2 dr^2 - \underbrace{2A(r)dt'\frac{d\Phi}{dr}dr} - \underbrace{B(r)rdt'dr} + B(r)\frac{d\Phi}{dr}r^2dr.$$

So if we set

$$\frac{d\Phi}{dr} = \frac{rB(r)}{2A(r)} \quad \text{our metric will become diagonal ,}$$

$$ds^2 = A(r)dt'^2 - \left[C(r) - \frac{1}{4}\frac{B^2(r)}{A(r)}\right] r^2 dr^2 - D(r)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right].$$

If we redefine the radius r

$$r'^2 = D(r)r^2$$

then we can write the metric as

$$ds^2 = \alpha(r')dt'^2 - \beta(r')dr'^2 - [r'^2 d\theta^2 + r'^2 \sin^2\theta d\phi^2],$$

and there is no need to retain the primes on the variables so we can rewrite the metric as

$$ds^2 = \alpha(r)dt^2 - \beta(r)dr^2 - [r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2]. \tag{3}$$

The general static, isotropic, metric is thus specified by two functions functions, $\alpha(r)$ and $\beta(r)$ which we need to determine by solving the Einstein field equations.

The components of the metric are

$$\begin{array}{lllll} g_{00} & = & g_{tt} & = & \alpha(r) \\ g_{11} & = & g_{rr} & = & -\beta(r) \\ g_{22} & = & g_{\theta\theta} & = & -r^2 \\ g_{33} & = & g_{\phi\phi} & = & -r^2\sin^2\theta \end{array} \qquad \begin{array}{lllll} g^{00} & = & g^{tt} & = & 1/\alpha(r) \\ g^{11} & = & g^{rr} & = & -1/\beta(r) \\ g^{22} & = & g^{\theta\theta} & = & -1/r^2 \\ g^{33} & = & g^{\phi\phi} & = & -\frac{1}{r^2\sin^2\theta} \end{array}$$

The affine connection is, by definition,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)$$

and so the components are:

$$\begin{aligned}
\Gamma^0_{00} &= 0, \\
\Gamma^i_{00} &= \frac{1}{2} g^{i\rho}\left\{\frac{-\partial g_{00}}{\partial x^\rho}\right\} \Longrightarrow \Gamma^r_{00} = -\frac{1}{2} g^{rr}\frac{\partial g_{00}}{\partial x^r} = \frac{1}{2}\frac{1}{\beta(r)}\frac{d\alpha(r)}{dr}, \\
\Gamma^0_{0i} &= \frac{1}{2} g^{0\rho}\left\{\frac{\partial g_{\rho 0}}{\partial x^i} + \frac{\partial g_{\rho i}}{\partial x^0} - \frac{\partial g_{0i}}{\partial x^\rho}\right\} \\
&= \frac{1}{2} g^{00}\left\{\frac{\partial g_{00}}{\partial x^i}\right\} \Longrightarrow \Gamma^0_{0r} = \frac{1}{2}\frac{1}{\alpha(r)}\frac{d\alpha(r)}{dr}, \\
\Gamma^0_{ij} &= 0, \\
\Gamma^i_{ii} &= \frac{1}{2} g^{i\rho}\left\{\frac{\partial g_{\rho i}}{\partial x^i} + \frac{\partial g_{\rho i}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^\rho}\right\} \\
&= \frac{1}{2} g^{ii}\left\{\frac{\partial g_{ii}}{\partial x^i}\right\} \Rightarrow \Gamma^r_{rr} = \frac{1}{2}\frac{1}{\beta(r)}\frac{d\beta(r)}{dr}, \\
\Gamma^r_{\theta\theta} &= \frac{1}{2} g^{rr}\left\{\frac{\partial g_{r\theta}}{\partial x^\theta} + \frac{\partial g_{r\theta}}{\partial x^\theta} - \frac{\partial g_{\theta\theta}}{\partial x^r}\right\} \\
&= -\frac{1}{2}\frac{1}{\beta(r)} 2r \Longrightarrow \Gamma^r_{\theta\theta} = -\frac{r}{\beta(r)},
\end{aligned}$$

$$
\begin{aligned}
\Gamma^{r}_{\phi\phi} &= -\frac{1}{2}g^{rr}\frac{\partial g_{\phi\phi}}{\partial r} = -\frac{r\sin^2\theta}{\beta(r)},\\
\Gamma^{\theta}_{\theta r} &= \frac{1}{2}g^{\theta\theta}\left\{\frac{\partial g_{\theta\theta}}{\partial x^r}\right\} = \frac{1}{r},\\
\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2}g^{\theta\theta}\left\{-\frac{\partial g_{\phi\phi}}{\partial x^\theta}\right\} = -\sin\theta\cos\theta,\\
\Gamma^{\phi}_{\phi r} &= \frac{1}{2}g^{\phi\phi}\left\{\frac{\partial g_{\phi\phi}}{\partial x^r}\right\} = \frac{1}{2}(-)\frac{1}{r^2\sin^2\theta}(-)2r\sin^2\theta = \frac{1}{r},\\
\Gamma^{\phi}_{\phi\theta} &= \frac{1}{2}g^{\phi\phi}\left\{\frac{\partial g_{\phi\phi}}{\partial x^\theta}\right\} = \frac{1}{2}(-)\frac{1}{r^2\sin^2\theta}(-)r^2\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta}.
\end{aligned}
$$

Summarising, we end up with nine non-vanishing components of Γ:

$$
\begin{aligned}
&(1) \qquad & \Gamma^{r}_{rr} &= \frac{\beta'}{2\beta},\\
&(2) \qquad & \Gamma^{r}_{\phi\phi} &= \frac{-r\sin^2\theta}{\beta},\\
&(3) \qquad & \Gamma^{\theta}_{r\theta} &= \frac{1}{r}\\
&(4) \qquad & \Gamma^{\phi}_{\phi r} &= \frac{1}{r}\\
&(5) \qquad & \Gamma^{0}_{0r} &= \frac{\alpha'}{2\alpha}\\
&(6) \qquad & \Gamma^{r}_{\theta\theta} &= -\frac{r}{\beta}\\
&(7) \qquad & \Gamma^{r}_{00} &= \frac{\alpha'}{2\beta}\\
&(8) \qquad & \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta\\
&(9) \qquad & \Gamma^{\phi}_{\phi\theta} &= \frac{\cos\theta}{\sin\theta},
\end{aligned}
$$

where the primes on α and β denote derivatives with respect to r,

$$
\beta' = \frac{d\beta}{dr}, \qquad \alpha' = \frac{d\alpha}{dr}.
$$

The Ricci tensor is, by definition,

$$
R_{\mu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^\lambda} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta}.
$$

It requires quite a lot of tedious (but simple) algebra to work out the components of the Ricci tensor. I will work out one of these components in full so that you can see what is involved and then state the results for the others:

$$
R_{rr} = \frac{\partial \Gamma^{\lambda}_{r\lambda}}{\partial x^r} - \frac{\partial \Gamma^{\lambda}_{rr}}{\partial x^\lambda} + \Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta} - \Gamma^{\eta}_{rr}\Gamma^{\lambda}_{\lambda\eta}.
$$

$$\begin{aligned}
\text{Term [1]} \quad & \frac{\partial\Gamma^{\lambda}_{r\lambda}}{\partial x^r} = \frac{\partial\Gamma^{r}_{rr}}{\partial r} + \frac{\partial\Gamma^{\theta}_{r\theta}}{\partial r} + \frac{\partial\Gamma^{\phi}_{r\phi}}{\partial r} + \frac{\partial\Gamma^{0}_{r0}}{\partial r}, \\
\text{Term [2]} \quad & \frac{\partial\Gamma^{\lambda}_{rr}}{\partial x^\lambda} = \frac{\partial\Gamma^{r}_{rr}}{\partial r}, \\
\text{Term [3]} \quad & \Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta} = \Gamma^{0}_{r\lambda}\Gamma^{\lambda}_{r0} + \Gamma^{i}_{r\lambda}\Gamma^{\lambda}_{ri} \\
& = \left(\Gamma^{0}_{r0}\right)^2 + \Gamma^{r}_{r\lambda}\Gamma^{\lambda}_{rr} + \Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{r\theta} + \Gamma^{\phi}_{r\lambda}\Gamma^{\lambda}_{r\phi} \\
& = (\Gamma^{0}_{r0})^2 + (\Gamma^{r}_{rr})^2 + (\Gamma^{\theta}_{r\theta})^2 + (\Gamma^{\phi}_{r\phi})^2, \\
\text{Term [4]} \quad & \Gamma^{\eta}_{rr}\Gamma^{\lambda}_{\lambda\eta} = \Gamma^{r}_{rr}\Gamma^{\lambda}_{\lambda r} = \Gamma^{r}_{rr}\left[\Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r} + \Gamma^{0}_{0r}\right].
\end{aligned}$$

Hence, summing these terms

$$R_{rr} = \frac{\partial\Gamma^{0}_{0r}}{\partial r} + \frac{\partial\Gamma^{\theta}_{\theta r}}{\partial r} + \frac{\partial\Gamma^{\phi}_{\phi r}}{\partial r} + \left(\Gamma^{0}_{0r}\right)^2 + \left(\Gamma^{r}_{rr}\right)^2 + \left(\Gamma^{\theta}_{r\theta}\right)^2 + \left(\Gamma^{\phi}_{r\phi}\right)^2 - \Gamma^{r}_{rr}\left[\Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r} + \Gamma^{0}_{0r}\right],$$

and so evaluating this expression using the components of the affine connection that we worked out before

$$R_{rr} = -\frac{1}{2}\frac{\alpha'^2}{\alpha^2} + \frac{1}{2}\frac{\alpha''}{\alpha} - \frac{2}{r^2} + \frac{1}{4}\frac{\alpha'^2}{\alpha^2} + \frac{1}{4}\frac{\beta'^2}{\beta^2} + \frac{1}{r^2} + \frac{1}{r^2} - \frac{\beta'}{2\beta}\left[\frac{\beta'}{2\beta} + \frac{1}{r} + \frac{\alpha'}{2\alpha} + \frac{1}{r}\right],$$

Hence,

$$R_{rr} = \frac{1}{2}\frac{\alpha''}{\alpha} - \frac{\alpha'}{4\alpha}\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) - \frac{1}{r}\frac{\beta'}{\beta}, \qquad (4a)$$

and, in a similar way, one can show that the rest of the components of the Ricci tensor are,

$$\begin{aligned}
R_{\theta\theta} &= -1 - \frac{r}{2\beta}\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right) + \frac{1}{\beta}, & (4b) \\
R_{\phi\phi} &= \sin^2\theta\, R_{\theta\theta}, & (4c) \\
R_{00} &= -\frac{\alpha''}{2\beta} + \frac{1}{4}\left(\frac{\alpha'}{\beta}\right)\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) - \frac{1}{r}\left(\frac{\alpha'}{\beta}\right), & (4d) \\
R_{\mu\nu} &= 0, \quad \mu \neq \nu. & (4e)
\end{aligned}$$

We showed in the last lecture that Einstein's field equations in vacuum are

$$R_{\mu\nu} = 0.$$

Hence *all* of the components of the Ricci tensor (4a) to (4d) *must vanish.* First, let's eliminate α'' from (4a) and (4d),

$$\frac{R_{rr}}{\beta} + \frac{R_{00}}{\alpha} = -\frac{\alpha'}{4\alpha\beta}\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) + \frac{\alpha'}{4\alpha\beta}\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) - \frac{1}{r\beta}\left(\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha}\right) = 0,$$

i.e.,

$$\frac{\beta'}{\beta} + \frac{\alpha'}{\alpha} = 0,$$

which requires that

$$\alpha(r)\beta(r) = \text{constant}.$$

Recall that the functions α and β appear in the metric as

$$ds^2 = \alpha(r)dt^2 - \beta(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

and so if we want to match on to the Minkowski metric as $r \to \infty$, the functions α and β must satisfy

$$\left.\begin{array}{l}\alpha(r) \to 1 \\ \beta(r) \to 1\end{array}\right\} \text{ as } r \to \infty, \quad \text{therefore } \alpha(r) = 1/\beta(r).$$

Now look at the other components:

$$\begin{aligned} R_{\theta\theta} &= -1 - \frac{r}{2}\alpha\left(-2\frac{\alpha'}{\alpha}\right) + \alpha \\ &= -1 + r\alpha' + \alpha = 0, \qquad \text{so} \quad \frac{d\left(r\alpha(r)\right)}{dr} = 1, \qquad i.e. \;\; r\alpha(r) = r + \text{constant}. \end{aligned} \tag{5}$$

$$R_{rr} = \frac{1}{2}\frac{\alpha''}{\alpha} + \frac{1}{r}\,\frac{\alpha'}{\alpha} = \frac{(r\alpha'' + 2\alpha')}{2r\alpha} = \frac{1}{2r\alpha}\frac{dR_{\theta\theta}}{dr},$$

and so setting $R_{\theta\theta} = 0$ over all space guarantees that $R_{rr} = 0$. Now, in the weak field limit we know that

$$\alpha(r) \to 1 + \underbrace{2\phi}_{\nearrow \atop \textit{Newtonian potential}}.$$

So, a long way from a mass M

$$\alpha(r) \to 1 - \frac{2GM}{r}$$

hence the constant in (5) must be $\underline{-2GM}$ and so

$$\alpha(r) = 1 - 2\frac{GM}{r}$$

Our metric is therefore

$$ds^2 = \left[1 - \frac{2GM}{r}\right]dt^2 - \left[1 - \frac{2GM}{r}\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{6}$$

This is called the *Schwarzchild* metric after its discoverer. This metric is important in General Relativity. It is an *exact* solution of the field equations and represents the metric around a spherical compact object of mass M. By construction, the metric reproduces the Newtonian potential $\phi = -GM/r$. However, since the metric (6) is an exact solution of the field equations, it is valid in the *strong* gravitational field limit $GM/r \gg 1$. We can therefore use the metric (6) to describe the orbits of planets around the Sun (a weak field application of GR) or to describe the orbits of particles around a compact object such as a black hole (a strong field application of GR). We will investigate both of these problems in the next few lectures.