Lecture Fourteen: Theory of Relativity

14.1 Equations of motion for the Schwarzchild metric

In the previous lecture, I derived the Schwarzchild metric

$$ds^2 = \overbrace{\left[1 - \frac{2GM}{r}\right]}^{\alpha(r)} dt^2 - \overbrace{\left[1 - \frac{2GM}{r}\right]^{-1}}^{\beta(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{1}$$

In this lecture, I will derive the equations of motion for a freely falling particle in the metric (1). The geodesic equations of motion are

$$\frac{d^2 x^\mu}{dp^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{dp} \frac{dx^\lambda}{dp} = 0, \tag{2}$$

where p is an affine parameter that varies along the path. (For massive particles we can normalize p so that $p = \tau$, but for the moment I will treat p as an arbitrary affine term. This will allow us to analyse the motion of photons). We derived the components of the affine connection for the Schwarzchild metric in the previous lecture, so it is straightforward to substitute these into (2) and derive the equations of motion for a freely falling particle. The equations of motion are:

$$\frac{d^2 x^0}{dp^2} + \Gamma^0_{\nu\lambda} \frac{dx^\nu}{dp} \frac{dx^\lambda}{dp} = \frac{d^2 x^0}{dp^2} + 2\Gamma^0_{0r} \frac{dx^0}{dp} \frac{dx^r}{dp},$$

So,
$$\boxed{\frac{d^2 t}{dp^2} + \frac{\alpha'}{\alpha} \frac{dt}{dp} \frac{dr}{dp} = 0} \tag{3a}$$

$$\frac{d^2 r}{dp^2} + \Gamma^r_{\nu\lambda} \frac{dx^\nu}{dp} \frac{dx^\lambda}{dp} = \frac{d^2 r}{dp^2} + \Gamma^r_{rr} \left(\frac{dr}{dp}\right)^2 + \Gamma^r_{\theta\theta} \left(\frac{d\theta}{dp}\right)^2 + \Gamma^r_{\phi\phi} \left(\frac{d\phi}{dp}\right)^2 + \Gamma^r_{00} \left(\frac{dt}{dp}\right)^2 = 0,$$

So,
$$\boxed{\frac{d^2 r}{dp^2} + \frac{1}{2}\frac{\beta'}{\beta} \left(\frac{dr}{dp}\right)^2 - \frac{r}{\beta} \left(\frac{d\theta}{dp}\right)^2 - \frac{r\sin^2\theta}{\beta} \left(\frac{d\phi}{dp}\right)^2 + \frac{1}{2}\frac{\alpha'}{\beta} \left(\frac{dt}{dp}\right)^2 = 0} \tag{3b}$$

$$\frac{d^2 \theta}{dp^2} + \Gamma^\theta_{\nu\lambda} \frac{dx^\nu}{dp} \frac{dx^\lambda}{dp} = \frac{d^2 \theta}{dp^2} + 2\Gamma^\theta_{r\theta} \left(\frac{dr}{dp}\right) \left(\frac{d\theta}{dp}\right) + \Gamma^\theta_{\phi\phi} \left(\frac{d\phi}{dp}\right)^2 = 0,$$

So,
$$\boxed{\frac{d^2 \theta}{dp^2} + \frac{2}{r} \frac{d\theta}{dp} \frac{dr}{dp} - \sin\theta\cos\theta \left(\frac{d\phi}{dp}\right)^2 = 0.} \tag{3c}$$

$$\frac{d^2\phi}{dp^2} + \Gamma^{\phi}_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = \frac{d^2\phi}{dp^2} + 2\Gamma^{\phi}_{\phi r}\frac{d\phi}{dp}\frac{dr}{dp} + 2\Gamma^{\phi}_{\phi\theta}\frac{d\phi}{dp}\frac{d\theta}{dp} = 0.$$

So,
$$\boxed{\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{d\phi}{dp}\frac{dr}{dp} + \frac{2\cos\theta}{\sin\theta}\frac{d\phi}{dp}\frac{d\theta}{dp} = 0.} \qquad (3d)$$

Since the field is isotropic, we can take the orbit to lie in the equatorial plane without loss of generality, *i.e.*, $\theta = \pi/2$. Equation (3c) becomes

$$\frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{d\theta}{dp}\frac{dr}{dp} = 0.$$

and so is satisfied if $\theta =$ constant. The equations of motion simplify to

$$\frac{d^2r}{dp^2} + \frac{1}{2}\frac{\beta'}{\beta}\left(\frac{dr}{dp}\right)^2 - \frac{r}{\beta}\left(\frac{d\phi}{dp}\right)^2 + \frac{1}{2}\frac{\alpha'}{\beta}\left(\frac{dt}{dp}\right)^2 = 0, \qquad (4a)$$

$$\frac{d^2t}{dp^2} + \frac{\alpha'}{\alpha}\frac{dt}{dp}\frac{dr}{dp} = 0, \qquad (4b)$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{d\phi}{dp}\frac{dr}{dp} = 0. \qquad (4c)$$

Look at equation (4c):

$$\text{write } \dot{\phi} = \frac{d\phi}{dp},$$

then (4c) becomes

$$\frac{1}{\dot{\phi}}\frac{d\dot{\phi}}{dp} + \frac{2}{r}\frac{dr}{dp} = 0,$$

$$i.e. \quad \frac{d\left[\ln\dot{\phi} + \ln r^2\right]}{dp} = 0.$$

Now look at (4b):

$$\text{write } \dot{t} = dt/dp,$$

then (4b) becomes

$$\frac{1}{\dot{t}}\frac{d\dot{t}}{dp} + \frac{1}{\alpha}\frac{d\alpha}{dp} = 0,$$

$$i.e. \quad \frac{d\left[\ln\dot{t} + \ln\alpha\right]}{dp} = 0.$$

So from these equations,

$$r^2\dot{\phi} = r^2\frac{d\phi}{dp} = \text{constant},$$

$$\alpha\frac{dt}{dp} = \text{constant},$$

but, we can choose the normalisation of p so that

$$\alpha\frac{dt}{dp} = 1, \qquad i.e. \qquad dt = \frac{dp}{\left[1 - 2\underbrace{GM/r}_{\substack{\uparrow \\ \text{if this is small } p \approx t}}\right]} \tag{5a}$$

We can write the other constant of integration as

$$r^2\frac{d\phi}{dp} = \underbrace{J}_{\substack{\uparrow \\ \text{GR equivalent of specific angular momentum.}}} \tag{5b}$$

We are left with one equation of motion for the radial coordinate:

$$\frac{d^2r}{dp^2} + \frac{1}{2}\frac{\beta'}{\beta}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{\beta r^3} + \frac{\alpha'}{2\beta\alpha^2} = 0.$$

Multiply this equation by $2\beta(dr/dp)$

$$2\beta\frac{dr}{dp}\frac{d^2r}{dp^2} + \beta'\left(\frac{dr}{dp}\right)^3 - \frac{2J^2}{r^3}\left(\frac{dr}{dp}\right) + \frac{\alpha'}{\alpha^2}\,\frac{dr}{dp} = 0,$$

$$i.e. \quad \frac{d}{dp}\left(\beta\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{\alpha}\right) = 0.$$

Hence

$$\beta\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{\alpha} = -\underbrace{\mathrm{E}}_{\substack{\uparrow \\ \text{GR equivalent of specific energy.}}} = \text{constant.} \tag{5c}$$

From equation (5c) and using (5a) and (5b) we must have

$$\beta dr^2 + \overbrace{r^2 d\phi^2}^{J=r^2d\phi/dp} - \alpha dt^2 = -\mathrm{E}\; dp^2,$$

and so comparing with the Schwarzchild metric (1), we must have

$$d\tau^2 = \mathrm{E}dp^2$$

and so,

$$\mathrm{E} > 0 \quad \text{for massive particles,} \quad (ds^2 > 0),$$

$$\mathrm{E} = 0 \quad \text{for photons,} \qquad\qquad (ds^2 = 0).$$

Notice that in a weak gravitational field the equations of motion become,

$$\begin{aligned} r^2 \frac{d\phi}{dt} &= J, \\ \left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{(1-2\phi)} &= -\mathrm{E}, \end{aligned}$$

i.e.

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{J^2}{r^2} + \phi = \left(\frac{1-\mathrm{E}}{2}\right),$$

which is identical to the Newtonian equation with the specific energy replaced by $\left(\frac{1-E}{2}\right)$.

14.2 Deflection of light by the Sun

Let's start with the equations of motion that we derived in the previous section:

$$\begin{aligned} \beta\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{\alpha} &= -E, \\ r^2\frac{d\phi}{dp} &= J. \end{aligned}$$

Eliminate p from the first of these equations,

$$\frac{\beta}{r^4}J^2\left(\frac{dr}{d\phi}\right)^2 + \frac{J^2}{r^2} - \frac{1}{\alpha} = -E,$$

and integrating,

$$\phi = \pm\int \frac{\beta^{1/2}dr}{r^2\left[\frac{1}{J^2\alpha} - \frac{1}{r^2} - \frac{E}{J^2}\right]^{1/2}}. \tag{6a}$$

For a photon ($E = 0$) this simplifies to

$$\phi = \pm\int \frac{\beta^{1/2}dr}{r^2\left[\frac{1}{J^2\alpha} - \frac{1}{r^2}\right]^{1/2}}. \tag{6b}$$

I will now use equation (6b) to analyse the deflection of light by the Sun. The relevant angles and distances are defined in Figure 1. The length b is the *impact parameter* – this is the distance of closest approach for a straight line orbit, *i.e.* if the photon travelled in a straight line, undeflected by the gravity of the Sun. In fact, the photon will be deflected by the gravity of the Sun and so it will move along a curved path as shown in the diagram. The distance r_0 is the *distance of closest approach* and this differs from the impact parameter as you can see from the diagram.

At large distances, the photon will move along a straight line with a speed c. I will use relativistic units ($c = 1$), in which case the straight line orbit can be written

$$b = r\sin(\phi - \phi_\infty) \approx r(\phi - \phi_\infty). \tag{7}$$

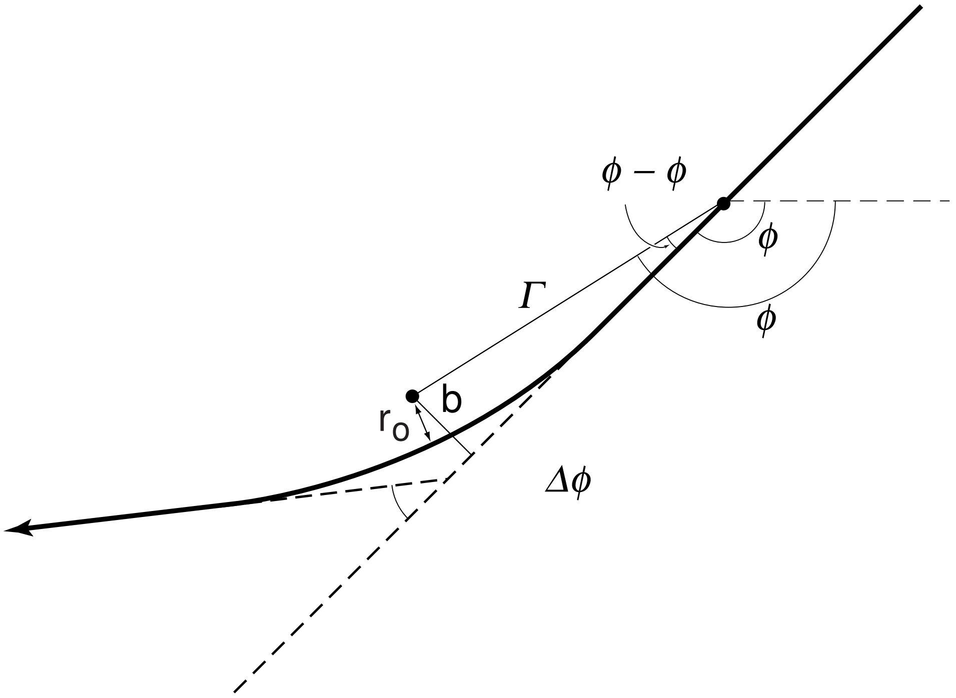

Figure 1: Angles and coordinates in the analysis of deflection of light by a point mass

Differentiating this equation with respect to time t,

$$\dot{r}\left(\phi-\phi_{\infty}\right)+r\dot{\phi}=0, \qquad \dot{r}=-1,$$

and so from equation (7)

$$r^2\dot{\phi}=b.$$

But recall that the specific angular momentum is defined by $r^2\dot{\phi}=J$, so for a photon the two integration constants of the orbit, E and J are,

$$\left.\begin{aligned} J&=b,\\ E&=0.\end{aligned}\right\}$$

At the point of closest approach, r_0,

$$\left(\frac{dr}{d\phi}\right)_{r_0}=0$$

so, from the equations of motion,

$$\frac{J^2}{r_0^2}=\frac{1}{\alpha(r_0)},$$

and so, the deflection angle is given by

$$\phi(r)=\phi_{\infty}+\int_r^{\infty}\frac{\beta^{1/2}dr}{r^2\left[\frac{\alpha(r_0)}{\alpha(r)}\frac{1}{r_0^2}-\frac{1}{r^2}\right]^{1/2}}, \tag{8}$$

where

$$
\begin{aligned}
\alpha(r) &= \left(1-\frac{2GM}{rc^2}\right), \\
\beta(r) &= \left(1-\frac{2GM}{rc^2}\right)^{-1}.
\end{aligned}
$$

The deflection from a straight line is

$$\triangle\phi = 2\left[\phi(r_0)-\phi_\infty\right]-\pi,$$

so all that remains is to evaluate the integral in equation (8). As it stands, equation (8) is a non-trivial integral, but we can reduce it to elementary integrals by expanding α and β in powers of $\frac{GM}{rc^2}$,

$$\int_r^\infty \frac{dr}{r}\frac{1}{\left(r^2/r_0^2-1\right)^{1/2}}\left[1+\frac{GM}{rc^2}+\frac{GM}{c^2r_0}\frac{r}{(r+r_0)}+\ldots\ldots\right],$$

Put $x=r/r_0$, $\gamma=\frac{GM}{r_0c^2}$, this integral becomes

$$\int_1^\infty \frac{dx}{x}\frac{1}{\left(x^2-1\right)^{1/2}}\left[1+\frac{\gamma}{x}+\gamma\frac{x}{(1+x)}+\mathcal{O}(\gamma^2)\ldots\right].$$

The first term integrates to $\pi/2$, and each of the next two terms integates to γ. [1] Hence the deflection angle is

$$\triangle\phi = 4\gamma = \frac{4GM}{r_0c^2},$$

and so for light grazing the Sun, General Relativity predicts a deflection of

$$\triangle\phi = 1.75''.$$

The 1919 eclipse expedition led by Eddington gave two sets of results

$$
\begin{aligned}
\triangle\phi &= 1.98\pm0.16'', \\
\triangle\phi &= 1.61\pm0.4'',
\end{aligned}
$$

consistent with the theory. This provided the first experimental verification of Einstein's theory and turned Einstein into a scientific superstar. (The media had a great story. Remember this was just after the end of the first world war, and so the headlines read something like 'Newton's theory of gravity overthrown by German physicist, verified by British scientists'). Some historians have argued that Eddington 'fiddled' the results to agree with the theory. If Eddington did indeed massage the results, then he gambled correctly. Modern radio experiments using VLBI (very long baseline interferometry) have been done to measure the gravitational deflection of the positions of radio quasars as they are eclipsed by the Sun. Such experiments can be done to an accuracy of better than $\sim 10^{-4}$ arcseconds. The following diagram is from the book by Clifford Will (*Theory and Experiment in Gravitational Physics*, CUP, 1981). It summarizes the results of VLBI measurements of the deflection angle $\Delta\phi$ from experiments conducted over the period $1969-1975$. The results are in excellent agreement with the predictions of General Relativity, and as you can see, they constrain the Brans-Dicke ω parameter to be $\omega \geq 40$.

[1] The integral of the third term is $\gamma\frac{(x-1)^{1/2}}{(x+1)^{1/2}}$.

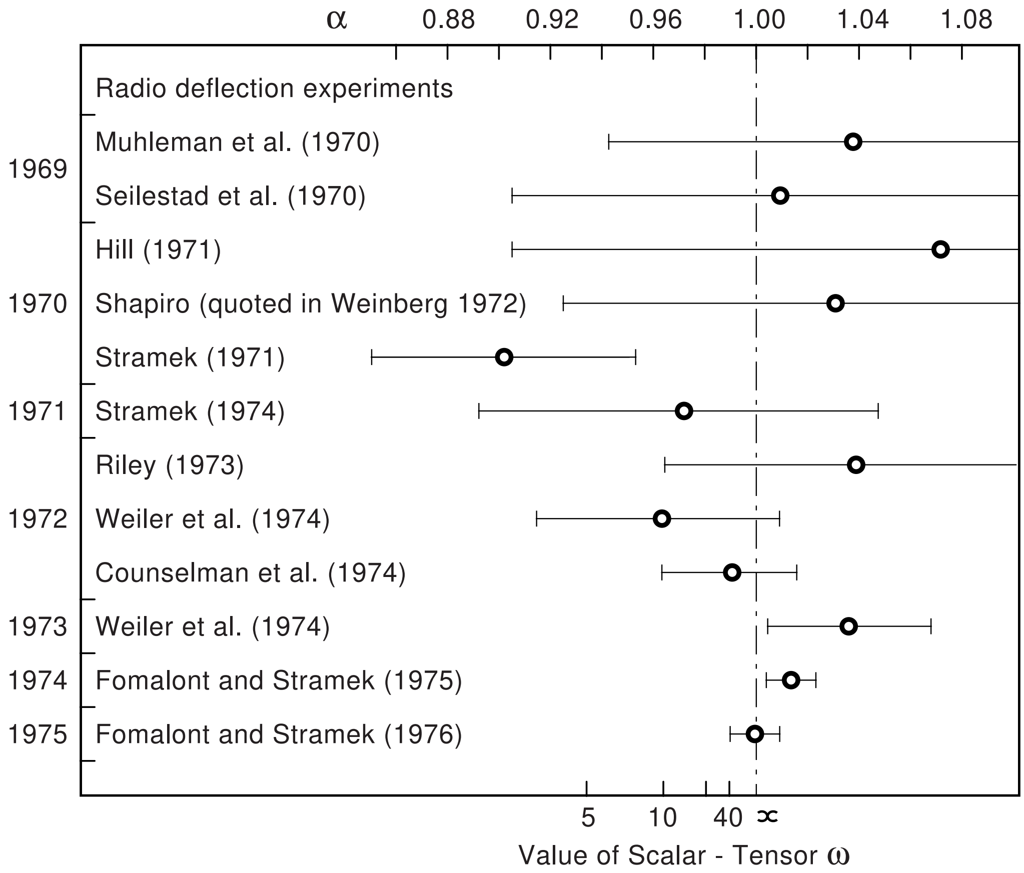

Figure 2: Results of radio-wave deflection measurements of the positions of quasars in the period 1969-1975. I have written the deflection angle as $\Delta\phi = \alpha(4GM/r_0c^2)$ and plotted the error bars on the parameter α. If GR is correct, we expect $\alpha = 1$.