Lecture Fifteen: Theory of Relativity

15.1 Precession of Planetary Orbits

In the preceeding section we derived the equation of motion

$$\frac{\beta}{r^4}J^2\left(\frac{dr}{d\phi}\right)^2+\frac{J^2}{r^2}-\frac{1}{\alpha}=-E. \tag{1}$$

Now, make the substitution $u=1/r$ and write $\mu=\frac{GM}{c^2}$ in (1). This gives

$$\left(\frac{du}{d\phi}\right)^2+u^2=-\frac{E}{J^2}(1-2\mu u)+2\mu u^3+\frac{1}{J^2}. \tag{2}$$

Now recall that

$$r^2\frac{d\phi}{d\tau}=\frac{J}{\sqrt{E}}=h=\text{constant},$$

and so differentiating (2) with respect to ϕ we get the simple equation of motion

$$\frac{d^2u}{d\phi^2}+u=\frac{\mu}{h^2}+3\mu u^2. \tag{3}$$

This equation gives the exact General Relativistic equation of motion for a particle around a point mass M. How does this equation compare with the Newtonian equation of motion? Assume, without loss of generality, that the orbit is in the plane

$$\theta=\text{constant}=\pi/2.$$

The Lagrangian of a point mass m is

$$\mathcal{L}=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)+\frac{GMm}{r},$$

and so from Lagrange's equations we can readily derive the the equations of motion:

$$r^2\dot{\phi}=h, \tag{4a}$$

$$\ddot{r}=\frac{h}{r^3}-\frac{GM}{r^2}. \tag{4b}$$

You will recognise the integration constant h in (4a) as the specific angular momentum of the particle. Now substitute $u=1/r$ and eliminate the time variable in equation (4b). The Newtonian equation of motion is

$$\frac{d^2u}{d\phi^2}+u=\frac{\mu}{h^2}. \tag{5}$$

Compare this with the General Relativistic equation of motion (3). It is *identical* except for the extra term $3\mu u^2$ on the right hand side. In fact we can write the GR equation as

$$\begin{aligned}\frac{d^2u}{d\phi^2}+u &= \frac{\mu}{h^2}+3\mu u^2\\ &= \frac{\mu}{h^2}\left[1+\frac{3h^2}{r^2c^2}\right]\\ &= \frac{\mu}{h^2}\left[1+\frac{3r^2}{c^2}\left(\frac{d\phi}{d\tau}\right)^2\right].\end{aligned}$$

Now, if the gravitational field is *weak* $\tau \approx t$ and we can replace $d\phi/d\tau$ by $d\phi/dt$. But $rd\phi/dt$ is just the *transverse* velocity, v_t. For example, if we consider the orbit of the Earth around the Sun, the transverse velocity is $v_t = 30\text{km/s}$. The difference between the Newtonian and General Relativistic equations of motion of the Earth around the Sun therefore differ by a tiny amount

$$\frac{\mu}{h^2}\left[1+\frac{3r^2}{c^2}\left(\frac{d\phi}{d\tau}\right)^2\right] = \frac{\mu}{h^2}\left[1+\frac{3v_t^2}{c^2}\right] \approx \frac{\mu}{h^2}\left[1+\mathcal{O}\left(3\times 10^{-8}\right)\right].$$

It should, of course, be no surprise that the deviation from the Newtonian equation of motion is so small. Newtonian gravity is a very good approximation for the orbits of planets – after all, that is how the inverse square law of Newtonian gravity was first deduced! General Relativity will therefore produce small deviations from the Newtonian orbits in the weak gravitational regime. But what about close to a massive object when the gravitational field is strong, $\mu u \approx 1$? In this regime we can derive a suprising result very simply.

15.1.1 Circular Orbits

In Newtonian mechanics you are used to thinking about circular orbits as a limiting case for particles with high angular momentum, *i.e.* when the velocity becomes purely tangential. What about circular orbits in General Relativity? For a circular orbit,

$$u = \text{constant},$$

and the equations of motion become,

$$\begin{aligned} u &= \frac{\mu}{h^2} + 3\mu u^2, \\ \frac{d\phi}{d\tau} &= hu^2. \end{aligned}$$

From the first of these equations

$$h^2\left(u - 3\mu u^2\right) = \mu,$$

and so from the second

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{\mu u^4}{(u-3\mu u^2)} = \frac{\mu}{r^2\left(r - 3GM/c^2\right)}.$$

What is the significane of the last equation? Evidently the last equation *cannot be satisfied* for circular orbits with $r < 3GM/c^2$! Such orbits cannot be geodesics (because they do not satisfy the geodesic equations) and so cannot be followed by freely falling particles. According to General Relativity, there are no stable circular orbits close to a massive body, no matter how large the angular momentum of a particle. This is very different from Newtonian theory.

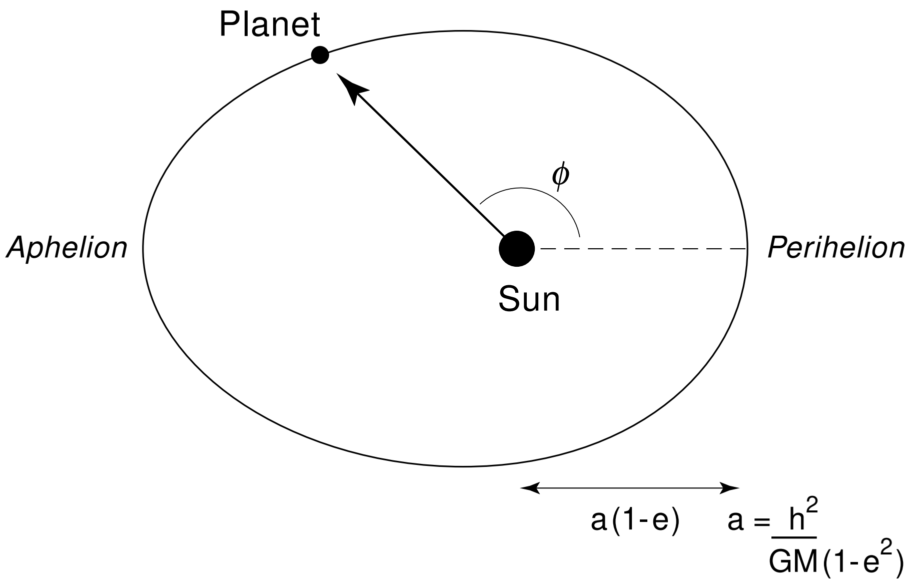

Figure 1: The elliptical orbit of a planet around the Sun.

15.1.2 Non-circular orbits

For a general non-circular orbit, Newton's equation

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{h^2},$$

has a solution

$$u = \frac{\mu}{h^2}(1 + e\cos\phi). \tag{6}$$

The last equation describes an ellipse, so for example, we can draw the orbit of a planet around the Sun as in the Figure 1.

The parameter e measures the ellipticity of the orbit. We can write the distance of closest approach (*perihelion*) as $r_1 = a(1-e)$ and the distance of furthest approach (*aphelion*) as $r_2 = a(1+e)$. The equation of motion then requires that

$$a = \frac{h^2}{GM(1-e^2)}.$$

If the gravitational field is weak, as it is for planetary orbits, then we expect Newtonian gravity to provide an excellent approximation to the motion in General Relativity. We can therefore treat the Newtonian solution (6) as the zeroth order solution to the GR equations of motion. Let's write the GR solution as

$$u = \frac{\mu}{h^2}(1 + e\cos\phi) + \Delta u,$$

where Δu is a perturbation. To first order in the perturbation Δu, the GR equation (3) gives

$$\frac{d^2\Delta u}{d\phi^2} + \Delta u = \frac{3\mu^3}{h^4}\left(1 + e^2\cos^2\phi + 2e\cos\phi\right),$$

which has the following particular integrals:

$$
\begin{aligned}
\Delta u_1 &= 3\mu^3/h^4, && \longleftarrow \text{tiny shift in orbit radius} \\
\Delta u_2 &= \tfrac{3}{2}\,\tfrac{\mu^3 e^2}{h^4} - \tfrac{1}{2}\,\tfrac{\mu^3}{h^4}\cos 2\phi, && \longleftarrow \text{tiny shift in orbit radius and oscillation} \\
\Delta u_3 &= \tfrac{3\mu^3}{h^4} e\phi \sin\phi, && \longleftarrow \text{cumulative change in shape of orbit}
\end{aligned}
$$

The first two terms Δu_1 and Δu_2 are immeasurably small, and are of no use in testing the theory. The third term Δu_3 is also small, but it is *cumulative.* It adds up over a long time interval to produce a measurable change in the shape of the orbit. To see this, add Δu_3 to the Newtonian solution:

$$
u = \frac{\mu}{h^2}\left[1 + e\left(\cos\phi + \underbrace{\frac{3\mu^2\phi}{h^2}}_{\substack{\ll 1 \text{ approximate as} \\ \sin\alpha,\quad \alpha \simeq \frac{3\mu^2\phi}{h^2}}} \sin\phi\right)\right],
$$

and using the relation

$$
\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha,
$$

we can write the solution as

$$
\begin{aligned}
u &\approx \frac{\mu}{h^2}\left[1 + e\cos(\phi - \alpha)\right] \\
&\approx \frac{\mu}{h^2}\left[1 + e\cos\left(\phi\left(1 - 3\mu^2/h^2\right)\right)\right],
\end{aligned}
$$

so the ellipse "closes" when

$$
\Delta\phi = \frac{2\pi}{(1 - 3\mu^2/h^2)} > 2\pi,
$$

i.e. we have to rotate the angle ϕ by *more* than 2π for the ellipse to "close". The result is that the ellipse cannot maintain a fixed shape, in other words the ellipse *precesses*:

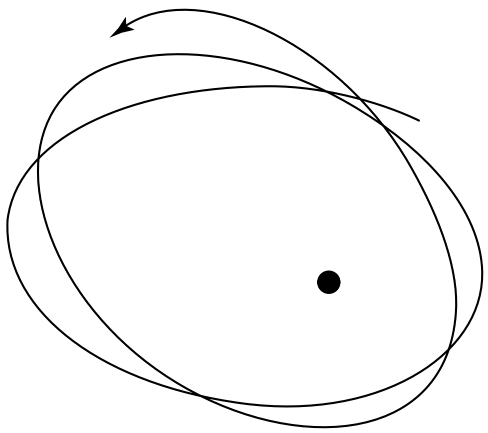

Figure 2: Precession of an elliptical orbit (greatly exaggerated).

In one revolution, the ellipse will rotate about the focus by an amount

$$\Delta = \frac{2\pi}{\left(1 - 3\frac{G^2M^2}{c^2h^2}\right)} - 2\pi \approx 6\pi\frac{G^2M^2}{c^2h^2} \approx \frac{6\pi GM}{a\left(1-e^2\right)c^2}, \tag{7}$$

where I have replaced h by the parameters that define the shape of the Newtonian orbit (*c.f.* Figure 1):

$$\begin{array}{ll} \text{Perihelion} & r_1 = a(1-e), \quad \phi = 0, \\ \text{Aphelion} & r_2 = a(1+e), \quad \phi = \pi, \end{array}$$

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2GM}{h^2} = \frac{2}{a\left(1-e^2\right).}$$

Now let's apply equation (7) to the orbit of Mercury, which has the followng parameters

$$\left.\begin{array}{lcl} \text{Period} & = & 88 \text{ days} \\ a & = & 5.8 \times 10^{10} \text{ m} \\ e & = & 0.2 \\ M_\odot & = & 2 \times 10^{30} \text{ kg} \end{array}\right\} \text{we find } \ \Delta = 43'' \ \text{ per century.}$$

In fact, the measured precession is

$$5599.7 \pm 0.4'' \ \text{ per century.}$$

but almost all of this is caused by perturbations from other planets. The residual, after taking perturbations into account, is in remarkable agreement with General Relativity. Here are the residuals for a number of planets (Icarus is a minor planet with a perihelion that lies within the orbit of Mercury):

	Observed Residual (arcsec per century)	Predicted Residual (arcsec per century)
Mercury	43.1 ± 0.5	43.03
Venus	8 ± 5	8.6
Earth	5 ± 1	3.8
Icarus	10 ± 1	10.3

In each case, the results are in excellent agreement with the predictions of GR. Einstein included this calculation in his 1915 paper on General Relativity. He had solved one of the major problems of celestial mechanics in the very first application of his complicated theory to an empirically testable problem. As you can imagine, this gave him tremendous confidence in his new theory.

15.2 Radar Echos

We have shown previously that photon paths obey the equation

$$\frac{1}{(1-2\mu/r)^3}\left(\frac{dr}{dt}\right)^2+\frac{J^2}{r^2}-\frac{1}{(1-2\mu/r)}=0, \tag{8}$$

where ($\mu \equiv GM/c^2$). (Use equations (5a) and (5c) from Section 14). Now consider a photon path from Earth to a planet (say Venus) as in the diagram below: Evidently, the photon

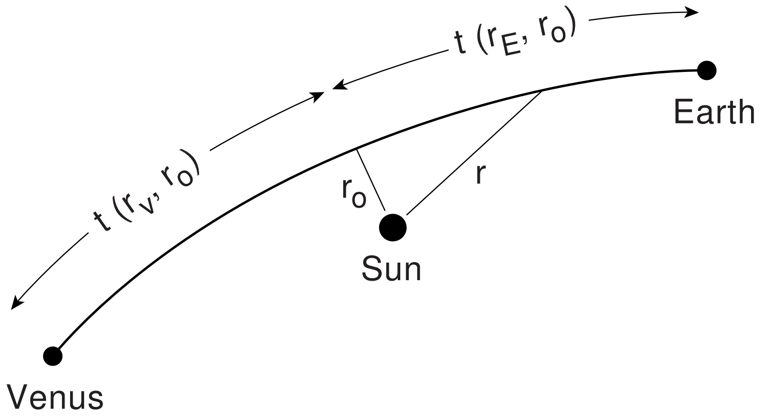

Figure 3: Photon path from Earth to Venus deflected by the Sun

path will be deflected by the gravitational field of the Sun. (I am, of course, assuming that the planets are in a configuration like that shown in Figure 3 where the photon has to pass close to the Sun to get to Venus). Let r_0 be the distance of closest approach of the photon to the Sun, then

$$\left(\frac{dr}{dt}\right)_{r_0}=0,$$

and so from (1), we can write the integration constant J as

$$J^2=\frac{r_0^2}{(1-2\mu/r_0)}.$$

We can therefore write the equation of motion (1) in the form

$$\frac{1}{(1-2\mu/r)^3}\left(\frac{dr}{dt}\right)^2+\frac{r_0^2}{r^2}\frac{1}{(1-2\mu/r_0)}-\frac{1}{(1-2\mu/r)}=0,$$

which we can integrate to give

$$t(r,r_0)=\int_{r_0}^{r}\frac{dr}{(1-2\mu/r)}\left[1-\frac{r_0^2}{r^2}\frac{(1-2\mu/r)}{(1-2\mu/r_0)}\right]^{-1/2}.$$

As in our derivation of the deflection of light by the Sun, expand this in powers of the small quantities μ/r and μ/r_0,

$$t\,(r,r_0)=\int_{r_0}^{r}\left(1-r_0^2/r^2\right)^{-1/2}\left[1+\frac{2\mu}{r}+\frac{\mu r_0}{(r+r_0)}\frac{1}{r}+\ldots\ldots\right],$$

which integrates to give

$$t(r, r_0) = \sqrt{r^2 - r_0^2} + 2\mu \ln\left[\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right] + \mu\left(\frac{r - r_0}{r + r_0}\right)^{1/2}.$$

The first term is just what we would have expected if light travelled in a straight line. The second and third term give us the extra time taken for the photon to travel along the *curved* path to the point r. So, you can see from Figure 3, that if we bounce a radar beam to Venus and back, then the excess time delay over a straight line path is

$$\Delta t = 2\left[t(r_E, r_0) + t(r_V,\ r_0) - \sqrt{r_E^2 - r_0^2} - \sqrt{r_V^2 - r_0^2}\right],$$

(where the factor of two is included because the photon has to get there and back). Since $r_E \gg r_0$, and $r_V \gg r_0$

$$t(r_E, r_0) - \sqrt{r_E^2 - r_0^2} \approx 2\mu \ln r_E/r_0 + \mu,$$

(and likewise for t_V and r_V), and so the excess time delay is

$$\Delta t \approx 4\mu\left[\ln\left(\frac{r_E r_V}{r_0^2}\right) + 1\right].$$

For Venus, when it is opposite to the Earth from the Sun

$$\Delta t \approx 220\ \mu s.$$

The idea of the experiment is as follows. Fire an intense radar beam towards Venus when it is almost opposite to us from Earth and measure the time delay of the radar echo with a sensitive radio telescope. The excess time delay gives us a test of the principle of equivalence. This sounds straightforward, but the time delay is very small and depends on the values of $r_E,\ r_V,\ r_0$. How can one determine these parameters to the required precision? The answer, is fit the entire time interval $t(r_E, r_0) + t(r_V, r_0)$ over a long period of time with the orbital parameters r_E, r_V, μ etc., as free parameters. The diagram on the next page shows an application of this test. There are a number of technical problems that limit the accuracy of this method. In practice, the radar beam is reflected off different points on the surface of Venus (mountain peaks, valleys, *etc*) and this introduces a dispersion in the time delay of several hundred μs. This problem can be solved by bouncing the radar beams from a mirror – as has since been done using the Viking Landers on Mars. Another, more complicated problem is correcting for refraction by the Solar corona – this can be important for photon paths which graze the surface of the Sun.